The Journal of Fuzzy Mathematics Vol. 15, No. 3, 2007
Los Angeles

Intuitionistic Fuzzy Relations over Intuitionistic Fuzzy Sets

Motilal Panigrahi* and Sudarsan Nanda

Department of Mathematics, Indian Institute of Technology,
Kharagpur-721302, West Bengal, INDIA
E-mail: motilal.panigrahi@gmail.com
snanda@maths.iitkgp.ernet.in

Abstract:

In this paper we have defined intuitionistic fuzzy relation from an intuitionistic fuzzy set to another intuitionistic fuzzy set. Defined some operations on these intuitinsitic fuzzy relations and studied some properties.

Key words:

Fuzzy sets, intuitionsitic fuzzy sets.

1. Introduction

Since the advent of fuzzy set theory by the pioneer Zadeh [10] in 1965, a lot of research has progressed in fuzzy relation. For a good overview of the theory of fuzzy relation, we refer the reader to Kaufmann [8]. After the introduction of fuzzy set theory many authors have generalized further and concepts like vague sets, rough sets and soft sets etc. have come. Atanassov [1,2,4] introduced the concept of intuitionitic fuzzy sets (IFS). He also developed intuitionistic fuzzy relation in [1]. Burillo-Bustince [6] have discussed more on intuitionitic fuzzy relations using t-norms and t-conorms. In classical set theory, a relation is defined between elements of two sets. Thus a relation in classical sense defines the “presence or absence” of a connection (or association) between the elements of two sets. A fuzzy relation [10] is usually defined as a fuzzy set on the Cartesian product of two sets. Thus for X and Y two ordinary sets a fuzzy relation R is defined as a fuzzy set on $X \times Y$. That is, R defines how strongly (or weakly) a pair $(a,b) \in X \times Y$ is related. However, if A and B are two fuzzy subsets of X (i.e. A and B are maps from X to [0,1]), then one can define a fuzzy relation between A and B (see Chakraborty-Das [7]).

Received July, 2005

* Research work is supported by AICTE, INDIA, and G. H. Patel College of Engg. And Tech., V V Nagar, Gujarat, INDIA
Correspondence to: Motilal Panigrahi, Dept of Maths, IIT Kharagpur-721302, INDIA.
1066-8950/01 $5.50

By taking motivations from [7], in this paper we have defined an intuitionsitic fuzzy relation between intuitionistic fuzzy subsets defined on a universal set. Based on this, in Section 3 we define some of the operations on intuitionsitic fuzzy relations and we study their properties. We prove (in Theorem 3.6, Theorem 3.9) that some results which are true in case of IFS are not true in case of intuitionistic fuzzy relations over intuitionistic fuzzy subsets. In Section 4 the composition of intuitionsitic fuzzy relations are defined. Elie Sanchez [9] has provided a methodology for certain basic fuzzy relational equations. We propose the corresponding problem of resolving an intuitionistic fuzzy relation equations. In Section 5 we have discussed the extension principle in this context. Finally in Section 6 we study the reflexivity, symmetry and transitivity in intuitionsitic fuzzy relations.

2. Preliminaries

Let X be a nonempty set. Then a fuzzy subset A of X is nothing but a function $\mu_A : X \to [0,1]$, called a membership function, whereas an *Intuitionistic Fuzzy Set (IFS)* A on X is an object of the from

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\} \tag{1}$$

where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ satisfy the rule

$$\mu_A(x) + \nu_A(x) \leq 1 \forall x \in X \tag{2}$$

called the *intuitionistic condition (IC).*

The numbers $\mu_A(x)$ and $\nu_A(x)$ are respectively called the degree of membership and degree ofnonmembership of the element x in the intuitionistic fuzzy set A. We denote the set of all intuitionistic fuzzy sets on X by $IFS(X)$ henceforth, if there is no confusion, an intuitionistic fuzzy set $A \in IFS(X)$ will be denoted as a pair (μ_A, ν_A).

Definition 2.1. For every two $IFSs$ [2] A and B on X, we define

(1) $A \subseteq B$ iff $(\forall x \in X)$ ($\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$),

(2) $A = B$ iff $A \subseteq B$ and $B \subseteq A$,

(3) $\overline{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in X\}$,

(4) $A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in X\}$,

(5) $A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in X\}$,

(6) $\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in X\}$,

(7) $\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle | x \in X\}$.

3. Intuitionistic fuzzy relations over intuitionistic fuzzy sets

Definition 3.1. Let X be the universal set and $A=(\mu_A,\nu_A)$, $B=(\mu_B,\nu_B)$ be two $IFSs$ of X. Define the Cartesian product $A\times B$ as the IFS of $X\times X$ by $A\times B=(\mu_{A\times B},\nu_{A\times B})$ where for all $x,y\in X$

$$\mu_{A\times B}(x,y):=\min(\mu_A(x),\mu_B(y)),\ \nu_{A\times B}(x,y):=\max(\nu_A(x),\nu_B(y))$$

Definition 3.2. Let R be an IFS of $X\times X$ with $R\subseteq A\times B$ i.e., $\forall(x,y)\in X\times X$ (i) $\mu_R(x,y)\le\mu_{A\times B}(x,y)$, (ii) $\nu_R(x,y)\ge\nu_{A\times B}(x,y)$ and (iii) $\mu_R(x,y)+\nu_R(x,y)\le 1$. Then we say that R is an intuitionsitic fuzzy relation from A to B. In particular, if $A=B$ then R is said to be an intuitionistic fuzzy relation on A.

We denote the set of all intuitionistic fuzzy relations from A to B by $IFR(A,B)$.

Definition 3.3. Let $R_1,R_2\in IFR(A,B)$. Then we say $R_1\subseteq R_2$ if for all $x,y\in X$, $\mu_{R_1}(x,y)\le\mu_{R_2}(x,y)$ and $\nu_{R_1}(x,y)\le\nu_{R_2}(x,y)$. If $R_1\subseteq R_2$ and $R_2\subseteq R_1$ then $R_1=R_2$.

Note that $R=A\times B$ is the strongest intuitionistic fuzzy relation from A to B. We define various operations on $IFR(A,B)$.

Definition 3.4. Let R, R_1, R_2 be intuitionistic fuzzy relations from A to B. Then $R_1\cup R_2$, $R_1\cap R_2$, R_1+R_2, $R_1\cdot R_2$, $R_1\Cup R_2$, $R_1\Cap R_2$, $\overline{R}$, R^{-1}, $\Box R$, $\Diamond R$, $R_1@R_2$, $R_1\$R_2$, $R_1\#R_2$, $R_1\star R_2$ are defined as follows:

(1) $R_1\cup R_2$:

$$\mu_{R_1\cup R_2}(x,y):=\max\left[\mu_{R_1}(x,y),\mu_{R_2}(x,y)\right],\ \nu_{R_1\cup R_2}(x,y):=\min\left[\nu_{R_1}(x,y),\nu_{R_2}(x,y)\right]$$

(2) $R_1\cap R_2$:

$$\mu_{R_1\cap R_2}(x,y):=\min\left[\mu_{R_1}(x,y),\mu_{R_2}(x,y)\right],\ \nu_{R_1\cap R_2}(x,y):=\max\left[\nu_{R_1}(x,y),\nu_{R_2}(x,y)\right]$$

(3) R_1+R_2:

$$\mu_{R_1+R_2}(x,y):=\mu_{R_1}(x,y)+\mu_{R_2}(x,y)-\mu_{R_1}(x,y)\cdot\mu_{R_2}(x,y),$$

$$\nu_{R_1+R_2}(x,y):=\nu_{R_1}(x,y)\cdot\nu_{R_2}(x,y)$$

(4) $R_1\cdot R_2$:

$$\mu_{R_1\cdot R_2}(x,y):=\mu_{R_1}(x,y)\cdot\mu_{R_2}(x,y),$$

$$\nu_{R_1+R_2}(x,y):=\nu_{R_1}(x,y)+\nu_{R_2}(x,y)-\nu_{R_1}(x,y)\cdot\nu_{R_2}(x,y)$$

(5) $R_1 \Cup R_2$:

$$\mu_{R_1 \Cup R_2}(x,y) := \min\left[1, \mu_{R_1}(x,y) + \mu_{R_2}(x,y)\right],$$

$$\nu_{R_1 \Cup R_2}(x,y) := \max\left[0, \nu_{R_1}(x,y) + \nu_{R_2}(x,y) - 1\right]$$

(6) $R_1 \Cap R_2$:

$$\mu_{R_1 \Cap R_2}(x,y) := \max\left[0, \mu_{R_1}(x,y) + \mu_{R_2}(x,y) - 1\right],$$

$$\nu_{R_1 \Cap R_2}(x,y) := \min\left[1, \nu_{R_1}(x,y) + \nu_{R_2}(x,y)\right]$$

(7) $\overline{R}$:

$$\mu_{\overline{R}}(x,y) := \min\left[\nu_R(x,y), \mu_{A\times B}(x,y)\right],\ \nu_{\overline{R}}(x,y) := \max\left[\mu_R(x,y), \nu_{A\times B}(x,y)\right]$$

(8) R^{-1}:

$$\mu_{R^{-1}}(x,y) := \mu_R(y,x),\ \nu_{R^{-1}}(x,y) := \nu_R(y,x).$$

(9) $\Box R$:

$$\mu_{\Box R}(x,y) := \mu_R(x,y),\ \nu_{\Box R}(x,y) := 1 - \mu_R(x,y).$$

(10) $\Diamond R$:

$$\mu_{\Diamond R}(x,y) := 1 - \nu_R(x,y),\ \nu_{\Diamond R}(x,y) := \nu_R(x,y).$$

(11) $R_1 @ R_2$:

$$\mu_{R_1 @ R_2}(x,y) := \left(\mu_{R_1}(x,y) + \mu_{R_2}(x,y)\right)/2,\ \nu_{R_1 @ R_2}(x,y) := \left(\nu_{R_1}(x,y) + \nu_{R_2}(x,y)\right)/2,$$

(12) $R_1 \$ R_2$:

$$\mu_{R_1 \$ R_2}(x,y) := \sqrt{\mu_{R_1}(x,y) \cdot \mu_{R_2}(x,y)},\ \nu_{R_1 \$ R_2}(x,y) := \sqrt{\nu_{R_1}(x,y) \cdot \nu_{R_2}(x,y)},$$

(13) $R_1 \# R_2$:

$$\mu_{R_1 \# R_2}(x,y) := \frac{2\mu_{R_1}(x,y) \cdot \mu_{R_2}(x,y)}{\left(\mu_{R_1}(x,y) + \mu_{R_2}(x,y)\right)},\ \nu_{R_1 \# R_2}(x,y) := \frac{2\nu_{R_1}(x,y) \cdot \nu_{R_2}(x,y)}{\left(\nu_{R_1}(x,y) + \nu_{R_2}(x,y)\right)}.$$

(In the last expression it is assumed that if $\mu_{R_1}(x,y) = \mu_{R_2}(x,y) = 0$, then $\mu_{R_1 \# R_2}(x,y) = 0$ and similarly for $\nu_{R_1 \# R_2}(x,y) = 0$ if $\nu_{R_1}(x,y) = \nu_{R_2}(x,y) = 0$.)

(14) $R_1 \star R_2$:

$$\mu_{R_1 \star R_2}(x,y) := \frac{\mu_{R_1}(x,y) + \mu_{R_2}(x,y)}{2\left(\mu_{R_1}(x,y)\cdot \mu_{R_2}(x,y) + 1\right)},\ \mu_{R_1 \star R_2}(x,y) := \frac{\nu_{R_1}(x,y) + \nu_{R_2}(x,y)}{2\left(\nu_{R_1}(x,y)\cdot \nu_{R_2}(x,y) + 1\right)}.$$

Theorem 3.5. *Among the operations defined in Definition 3.4, the following are closed, i.e., for* R_1, R_2, $R \in IFR(A,B)$, *we have* $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 \cdot R_2$, $R_1 \Cap R_2$, $\overline{R}$, $R_1 @ R_2$, $R_1 \$ R_2$, $R_1 \# R_2$ *all are intuitionistic fuzzy relations from* A *to* B. $R^{-1} \in IFR(B,A)$. *The operations* $+$, $\Cup$, $\star$ *are not closed, i.e.,* $R_1 + R_2$, $R_1 \Cup R_2$ *and* $R_1 \star R_2$ *may not belong to* $IFR(A,B)$.

Proof. It is easy to check that, for R_1, R_2, $R \in IFR(A,B)$, $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 \cdot R_2$, $R_1 \Cap R_2$, $\overline{R}$, $R_1 @ R_2$, $R_1 \$ R_2$, $R_1 \# R_2$ and are intuitionistic fuzzy relations from A to B.

We will show by examples that the operations $+$, $\Cup$, $\star$ are not closed.

Let $X = \{a,b,c\}$, $A = \{a|(0.3,0.6), b|(0.4,0.5), c|(0.7,0.2)\}$ and $B = \{a|(0.4,0.2), b|(0.5,0.4), c|(0.3,0.5)\}$, then

$A \times B =$

↱	a	b	c
a	$(0.3,0.6)$	$(0.3,0.6)$	$(0.3,0.6)$
b	$(0.4,0.5)$	$(0.4,0.5)$	$(0.3,0.5)$
c	$(0.4,0.2)$	$(0.5,0.4)$	$(0.3,0.5)$

Let

$R_1 =$

↱	a	b	c
a	$(0.2,0.7)$	$(0.2,0.8)$	$(0.1,0.9)$
b	$(0.3,0.6)$	$(0.3,0.7)$	$(0.2,0.7)$
c	$(0.3,0.5)$	$(0.2,0.7)$	$(0.2,0.6)$

and

$R_2 =$

↱	a	b	c
a	$(0.3,0.7)$	$(0.3,0.7)$	$(0.2,0.8)$
b	$(0.3,0.7)$	$(0.2,0.6)$	$(0.2,0.7)$
c	$(0.2,0.6)$	$(0.2,0.6)$	$(0.2,0.7)$

Now

(1)

$$\mu_{R_1+R_2}(a,a) = \mu_{R_1}(a,a) + \mu_{R_2}(a,a) - \mu_{R_1}(a,a)\cdot \mu_{R_2}(a,a) = 0.2 + 0.3 - (0.2)(0.3) = 0.44,$$

but $\mu_{A\times B}(a,a)=0.3$. Hence $\mu_{R_1+R_2}(a,a)\nless\mu_{A\times B}(a,a)$. Therefore $R_1+R_2\notin IFR(A,B)$.

(2) $\mu_{R_1\Cup R_2}(b,a)=\min\left[1,\mu_{R_1}(b,a)+\mu_{R_2}(b,a)\right]=\min[1,0.3+0.3]=0.6$, but $\mu_{A\times B}(b,a)=0.4$. Hence $\mu_{R_1\Cup R_2}(b,a)\nless\mu_{A\times B}(b,a)$. Therefore $R_1\Cup R_2\notin IFR(A,B)$.

(3) $\nu_{R_1\star R_2}(b,a)\coloneqq\nu_{R_1}(b,a)+\nu_{R_2}(b,a)/2\left(\nu_{R_1}(b,a)\cdot\nu_{R_2}(b,a)+1\right)=1.3/2.84$, but $\nu_{A\times B}(b,a)=0.5$. Hence $\nu_{R_1\star R_2}(b,a)\ngtr\nu_{A\times B}(b,a)$. Therefore $R_1\star R_2\notin IFR(A,B)$.

Theorem 3.6. *Let* $R\in IFR(A,B)$, *then* $R\subseteq\bar{R}$.

Proof. Let $R=(\mu_R,\nu_R)$. $\bar{R}=(\mu_{\bar{R}},\nu_{\bar{R}})$, where

$$\mu_{\bar{R}}(x,y)=\min\left(\nu_R(x,y),\mu_{A\times B}(x,y)\right),\ \nu_{\bar{R}}(x,y)=\max\left(\nu_R(x,y),\mu_{A\times B}(x,y)\right).$$

Now $\bar{\bar{R}}=\left(\mu_{\bar{\bar{R}}},\nu_{\bar{\bar{R}}}\right)$, where

$$\mu_{\bar{\bar{R}}}(x,y)=\min\left(\nu_{\bar{R}}(x,y),\mu_{A\times B}(x,y)\right)=\min\left(\max\left(\mu_R(x,y),\nu_{A\times B}(x,y)\right),\mu_{A\times B}(x,y)\right),$$

$$\nu_{\bar{\bar{R}}}(x,y)=\max\left(\mu_{\bar{R}}(x,y),\nu_{A\times B}(x,y)\right)=\max\left(\min\left(\nu_R(x,y),\mu_{A\times B}(x,y)\right),\nu_{A\times B}(x,y)\right).$$

We have to show that $\forall(x,y)\in X\times X$,

$$\mu_R(x,y)\le\min\left(\max\left(\mu_R(x,y),\nu_{A\times B}(x,y)\right),\mu_{A\times B}(x,y)\right),$$

$$\nu_R(x,y)\ge\max\left(\min\left(\nu_R(x,y),\mu_{A\times B}(x,y)\right),\nu_{A\times B}(x,y)\right). \tag{3}$$

Remembering that $\mu_R(x,y)\le\mu_{A\times B}(x,y)$, $\nu_R(x,y)\ge\nu_{A\times B}(x,y)$ and taking various orders among $\mu_R(x,y)$, $\mu_{A\times B}(x,y)$, $\nu_R(x,y)$ and $\nu_{A\times B}(x,y)$, we can easily see that inequalities (3) hold good. Hence $R\subseteq\bar{\bar{R}}$.

Note 3.7. For every $A\in IFS(X)$, we have $\Box A=\overline{\Box\bar{A}}$ and $\Diamond=\overline{\Box\bar{A}}$ (see Theorem 2 [2]). But for $A,B\in IFS(X)$ and $R\in IFR(A,B)$, as such $\Box R\ne\overline{\Box\bar{R}}$. For which we furnish the following example.

Example 3.8. Let $X=\{a,b,c\}$. Let $A,B\in IFS(X)$ and $R\in IFR(A,B)$ be given as in the following table:

	μ_A	ν_A	μ_B	ν_B	μ_R	ν_R	$\mu_{\Box R}$	$\nu_{\Box R}$	$\mu_{\overline{\Box\bar{R}}}$	$\nu_{\overline{\Diamond\bar{R}}}$

a	0.4	0.5	0.7	0.2	0.3	0.6	0.3	0.7	0.4	0.5
b	0.7	0.2	0.6	0.4	0.2	0.8	0.2	0.8	0.4	0.6
c	0.4	0.5	0.5	0.4	0.3	0.6	0.3	0.7	0.4	0.5

Thus it can be easily seen that $\Box R \neq \overline{\Diamond \overline{R}}$. However we have the following theorem.

Theorem 3.9. *Let* $R \in IFR(A,B)$, *then (1)* $\Box R \subseteq \overline{\Diamond \overline{R}}$ *and (2)* $\Diamond R \subseteq \overline{\Box \overline{R}}$.

Proof. (1) By definition $\Box R$ is given by $\mu_{\Box R}(x,y) = \mu_R(x,y)$, $\nu_{\Box R}(x,y) = 1 - \mu_R(x,y)$. Now $\overline{\Diamond \overline{R}} = \left(\mu_{\overline{\Diamond \overline{R}}}, \nu_{\overline{\Diamond \overline{R}}}\right)$ is given by

$$\mu_{\overline{\Diamond \overline{R}}}(x,y) = \min\left[\nu_{\Diamond \overline{R}}(x,y), \mu_{A\times B}(x,y)\right] = \min\left[\nu_{\overline{R}}(x,y), \mu_{A\times B}(x,y)\right]$$

$$= \min\left[\max\left(\mu_R(x,y), \nu_{A\times B}(x,y)\right), \mu_{A\times B}(x,y)\right]$$

and similarly $\nu_{\overline{\Diamond \overline{R}}}$ can be found.

First we will show that $\mu_{\Box R}(x,y) \leq \mu_{\overline{\Diamond \overline{R}}}(x,y) \quad \forall (x,y) \in X \times X$. Remembering that $\mu_R(x,y) \leq \mu_{A\times B}(x,y) \quad \forall (x,y) \in X \times X$, we have the following three possibilities:

(1) $\mu_R(x,y) \leq \nu_{A\times B}(x,y) \leq \mu_{A\times B}(x,y)$,

(2) $\mu_R(x,y) \leq \mu_{A\times B}(x,y) \leq \nu_{A\times B}(x,y)$,

(3) $\nu_{A\times B}(x,y) \leq \mu_R(x,y) \leq \mu_{A\times B}(x,y)$.

In each of the three cases,

$$\mu_{\Box R}(x,y) = \mu_R(x,y) \leq \mu_{\overline{\Diamond \overline{R}}}(x,y). \tag{4}$$

Also, $\nu_{\Box R}(x,y) = 1 - \mu_R(x,y) \geq 1 - \mu_{\overline{\Diamond \overline{R}}}(x,y) \geq \nu_{\overline{\Diamond \overline{R}}}(x,y)$ ($\because \quad \mu_{\overline{\Diamond \overline{R}}}(x,y) + \nu_{\overline{\Diamond \overline{R}}}(x,y) \leq 1$). Hence the proof of the part (1) of the theorem.

(2) The proof of (2) is similar to the part (1).

Theorem 3.10. *For* $A, B \in IFS(X)$, *we have*

(1) $\Box(A \times B) = \Box A \times \Box B$,

(2) $\Diamond(A \times B) = \Diamond A \times \Diamond B$

Proof. The proof is easy and we omit it. Also we have the following results.

Theorem 3.11. *For* $R_1, R_2 \in IFR(A,B)$ *we have*

(1) $\Box(R_1 \cup R_2) = \Box R_1 \cup \Box R_2$

(2) $\Diamond(R_1 \cup R_2) = \Diamond R_1 \cup \Diamond R_2$

(3) $\square(R_1 \cap R_2) = \square R_1 \cap \square R_2$

(4) $\lozenge(R_1 \cap R_2) = \lozenge R_1 \cap \lozenge R_2$

(5) $\square(R_1 + R_2) = \square R_1 + \square R_2$

(6) $\lozenge(R_1 + R_2) = \lozenge R_1 + \lozenge R_2$

(7) $\square(R_1 \cdot R_2) = \square R_1 \cdot \square R_2$

(8) $\lozenge(R_1 \cdot R_2) = \lozenge R_1 \cdot \lozenge R_2$

(9) $\square(R_1 \Cap R_2) = \square R_1 \Cap \square R_2$

(10) $\lozenge(R_1 \Cap R_2) = \lozenge R_1 \Cap \lozenge R_2$

(11) $\square(R_1 \Cup R_2) = \square R_1 \Cup \square R_2$

(12) $\lozenge(R_1 \Cup R_2) = \lozenge R_1 \Cup \lozenge R_2$

Theorem 3.12. *For* $R_1, R_2 \in IFR(A,B)$ *we have*

(1) $R_1 \subseteq R_2 \Leftrightarrow R_1^{-1} \subseteq R_2^{-1}$,

(2) $\left(R_1^{-1}\right)^{-1} = R_1$,

(3) $\left(R_1 * R_2\right)^{-1} = R_1^{-1} * R_2^{-1}$ *where* $*$ *stands for the operations* $\cup$, $\cap$, $+$, $\cdot$, $\Cap$, $\Cup$, $@$, $\$$, $\#$, $\star$.

(4) $(\square R)^{-1} = \square\left(R^{-1}\right)$.

(5) $(\lozenge R)^{-1} = \lozenge\left(R^{-1}\right)$.

Proof. (1) $R_1 \subset R_2 \Leftrightarrow \mu_{R_1}(x,y) \le \mu_{R_2}(x,y)$ and $\nu_{R_1}(x,y) \ge \nu_{R_2}(x,y) \Leftrightarrow \mu_{R_1^{-1}}(y,x) \le \mu_{R_2^{-1}}(y,x)$ and $\nu_{R_1^{-1}}(y,x) \ge \nu_{R_2^{-1}}(y,x) \Leftrightarrow R_1^{-1} \subseteq R_2^{-1}$.

(2) It is easy to prove and we omit it.

(3) We prove for one operation $\cup$, rest are similar and we omit the proofs.

$$\mu_{(R_1 \cup R_2)^{-1}}(x,y) = \mu_{R_1 \cup R_2}(y,x) = \max\left(\mu_{R_1}(y,x), \mu_{R_2}(y,x)\right)$$

$$= \max\left(\mu_{R_1^{-1}}(x,y), \mu_{R_2^{-1}}(x,y)\right) = \mu_{R_1^{-1} \cup R_2^{-1}}(x,y)$$

Similarly, $\nu_{(R_1 \cup R_2)^{-1}}(x,y) = \nu_{R_1^{-1} \cup R_2^{-1}}(x,y)$. Therefore, $(R_1 \cup R_2)^{-1} = R_1^{-1} \cup R_2^{-1}$.

(4) $\mu_{(\square R)^{-1}}(x,y) = \mu_{(\square R)}(y,x) = \mu_R(y,x) = \mu_{R^{-1}}(x,y) = \mu_{\square R^{-1}}(x,y)$,

$$\nu_{(\square R)^{-1}}(x,y) = \nu_{(\square R)}(y,x) = 1 - \mu_R(y,x) = 1 - \mu_{R^{-1}}(x,y) = \nu_{\square R^{-1}}(x,y).$$

(5) The proof is similar to above.

Definition 3.13 [2] For $A \in IFS(X)$, we use the following notations:

$$K_A = \max_{x\in X} \mu_A(x),\ L_A = \min_{x\in X} \nu_A(x),\ k_A = \min_{x\in X} \mu_A(x),\ l_A = \max_{x\in X} \nu_A(x)$$

$$C(A) = \{\langle x, K_A, L_A\rangle | x \in X\} \text{ and } I(A) = \{\langle x, k_A, l_A\rangle | x \in X\}$$

Definition 3.14. For $R \in IFR(A,B)$, we use the following notations:

$$K_R = \max_{(x,y)\in X\times X} \mu_R(x,y),\ L_R = \min_{(x,y)\in X\times X} \nu_R(x,y),$$

$$k_R = \min_{(x,y)\in X\times X} \mu_R(x,y),\ l_R = \max_{(x,y)\in X\times X} \nu_R(x,y)$$

$$C(R) = \{\langle (x,y), K_R, L_R\rangle | (x,y) \in X\times X\},$$

$$I(R) = \{\langle (x,y), k_R, l_R\rangle | (x,y) \in X\times X\}.$$

Theorem 3.15. *For $A, B \in IFS(X)$, we have the following*

(1) $C(R)$ *may not be an* IFR *from* A *to* B,

(2) $C(R)$ *is an* IFR *from* $C(A)$ *to* $C(B)$,

(3) $I(R)$ *is an* IFR *from* A *to* B,

(4) $I(R)$ *is an* IFR *from* $I(A)$ *to* $I(B)$,

(5) $\bigcup_{R\in IFR(A,B)} C(R) = C(A\times B) = C(A)\times C(B)$,

(6) $\bigcup_{R\in IFR(A,B)} I(R) = I(A\times B) = I(A)\times I(B)$.

Proof. (1) We show by means of an example that $C(R)$ may not be an IFR from A to B .Let $X = \{a,b,c\}$, $A = \{a|(0.3,0.6), b|(0.5,0.4), c|(0.8,0.2)\}$ and $B = \{a|(0.7,0.1), b|(0.6,0.2), c|(0.3,0.6)\}$, then

$A \times B =$

Γ	a	b	c
a	$(0.3,0.6)$	$(0.3,0.6)$	$(0.3,0.6)$
b	$(0.5,0.4)$	$(0.5,0.4)$	$(0.3,0.6)$
c	$(0.7,0.2)$	$(0.6,0.2)$	$(0.3,0.6)$

Let

$$R=\begin{array}{|c|c|c|c|}\hline \Gamma & a & b & c \\ \hline a & (0.2,0.7) & (0.2,0.8) & (0.1,0.7) \\ b & (0.2,0.5) & (0.1,0.5) & (0.2,0.8) \\ c & (0.2,0.4) & (0.5,0.5) & (0.1,0.9) \\ \hline \end{array}$$

Clearly, R is an IFR from A to B. Now $K_A=0.8$, $L_A=0.2$, $K_B=0.7$ and $L_B=0.1$ hence,

$$C(A)=\left\{a\left|(0.8,0.2),b\right|(0.8,0.2),c\left|(0.8,0.2)\right.\right\}$$

and

$$C(B)=\left\{a\left|(0.7,0.1),b\right|(0.7,0.1),c\left|(0.7,0.1)\right.\right\},$$

$$C(A\times B)=\left\{\left\langle(x,y),0.7,0.2\right\rangle\middle|(x,y)\in X\times X\right\},$$

and

$$C(R)=\left\{\left\langle(x,y),0.2,0.4\right\rangle\middle|(x,y)\in X\times X\right\}.$$

Note that $C(R)\not\subseteq A\times B$, since $\mu_{C(R)}(a,b)=0.2$, $\nu_{C(R)}(a,b)=0.4$, $\mu_{A\times B}(a,b)=0.3$ and $\nu_{A\times B}(a,b)=0.6$. Hence $C(R)$ is not an intuitionsitic fuzzy relation from A to B.

(2) By definition

$$C(R)=\left\{\left\langle(x,y),K_R,L_R\right\rangle\middle|(x,y)\in X\times X\right\},$$

where

$$K_R=\max_{(x,y)\in X\times X}\mu_R(x,y),\ L_R=\min_{(x,y)\in X\times X}\nu_R(x,y),$$

Since $\mu_R(x,y)\le\mu_{A\times B}(x,y)=\min(\mu_A(x),\mu_B(y))$, $\therefore$ $\mu_R(x,y)\le\mu_A(x)$ and $\mu_R(x,y)\le\mu_B(y)\Rightarrow\max_{(x,y)}\mu_R(x,y)\le\max_x\mu_A(x)$ and $\max_{(x,y)}\mu_R(x,y)\le\max_x\mu_B(y)$ $\Rightarrow K_R\le K_A$ and $K_R\le K_B\Rightarrow K_R\le\min(K_A,K_B)$. Similarly, $L_R\ge\max(L_A,L_B)$. Hence $C(R)$ is an IFR from $C(A)$ to $C(B)$.

(3) The proof is easy and we omit it.

(4) The proof is analogous to (2).

(5) $\bigcup_{R\in IFR(A,B)}C(R)=\left\{\left\langle(x,y),\sup_{R\in IFR(A,B)}K_R,\inf_{R\in IFR(A,B)}L_R\right\rangle\middle|(x,y)\in X\times X\right\}=$ $\left\{\left\langle(x,y),\sup_{R\in IFR(A,B)}\left(\max_{(a,b)\in X\times X}\mu_R(a,b)\right),\inf_{R\in IFR(A,B)}\left(\min_{(a,b)\in X\times X}\nu_R(a,b)\right)\right\rangle\middle|(x,y)\in X\times X\right\}$.

Now we will show that $\sup_{R\in IFR(A,B)}\left(\max_{(a,b)\in X\times X}\mu_R(a,b)\right)=K_{A\times B}=\min\left[K_A,K_B\right]$. Since $A\times B\in IFR(A,B)$, we have

$$\sup_{R\in IFR(A,B)}\left(\max_{(a,b)\in X\times X}\mu_R(a,b)\geq\max_{(a,b)\in X\times X}\mu_{A\times B}(a,b)\right)=K_{A\times B}.$$

Also since $\mu_R(a,b)\leq\mu_{A\times B}(a,b)\Rightarrow\max_{(a,b)\in X\times X}\mu_R(a,b)\leq\max_{(a,b)\in X\times X}\mu_{A\times B}(a,b)=K_{A\times B}$, now by taking sup on either side we get the desired result. Also we have

$$K_{A\times B}=\max_{(x,y)}\mu_{A\times B}(x,y)=\max_{(x,y)}\left[\min\left(\mu_A(x),\mu_B(y)\right)\right]$$

$$=\min\left[\max_{x\in X}\mu_A(x),\max_{y\in X}\mu_B(y)\right]=\min\left[K_A,K_B\right].$$

Similarly, we have

$$L_{A\times B}=\max\left[L_A,L_B\right].\qquad C(A\times B)=C(A)\times C(B).$$

(6) Proof is similar to the last one.

4. Composition of intuitionistic fuzzy relations

Definition 4.1. For $R_1\in IFR(A,B)$ and $R_2\in IFR(B,C)$ define the composition '$\circ$' by, $\forall x,y\in X$

$$\mu_{R_1\circ R_2}(x,y):=\max_{z\in X}\left[\min\left(\mu_{R_1}(x,z),\mu_{R_2}(z,y)\right)\right],$$

$$\nu_{R_1\circ R_2}(x,y):=\min_{z\in X}\left[\max\left(\nu_{R_1}(x,z),\nu_{R_2}(z,y)\right)\right].$$

As a particular case of composition, if P is an intuitionistic fuzzy set of X and $R\in IFR(A,B)$, then $S=P\circ R$ is an IFS of X defined as follows: $\forall y\in X$

$$\mu_{P\circ R}(y):=\max_{z\in X}\left[\min\left(\mu_P(z),\mu_R(z,y)\right)\right],$$

$$\nu_{P\circ R}(y):=\min_{z\in X}\left[\max\left(\nu_P(z),\nu_R(z,y)\right)\right].$$

Similarly $T=R\circ P$ is an IFS of X which can be defined as $\forall y\in X$

$$\mu_{R\circ P}(y):=\max_{z\in X}\left[\min\left(\mu_R(y,z),\mu_P(z)\right)\right],$$

$$\nu_{R\circ P}(y) := \min_{z\in X}\left[\max\left(\nu_R(y,z),\nu_P(z)\right)\right].$$

Theorem 4.2. *For $R_1 \in IFR(A,B)$ and $R_2 \in IFR(B,C)$ the composition $R_1 \circ R_2$ is an IFR from A to C.*

Proof. $\mu_{R_1\circ R_2}(x,y) \le \mu_{A\times B}(x,y)$ (see [7]). Similarly $\nu_{R_1\circ R_2}(x,y) \ge \nu_{A\times B}(x,y)$. It remains to show that $\mu_{R_1\circ R_2}(x,y) + \nu_{R_1\circ R_2}(x,y) \le 1$. But this is easy to prove so we omit it.

Note. Elie Sanchez [9] has provided a methodology for solution of certain basic fuzzy relational equations. Here we pose an open problem:

Problem. *Let X be the universal set. Let $A,B,C \in IFS(X)$. Let $R_1 \in IFR(A,B)$ and $R_2 \in IFR(B,C)$. Consider the following set*

$$Y = \left\{R \in IFR(B,C) : R_1 \circ R = R_2\right\}$$

If Y is nonempty, then it is desirable to have a method to find an $R \in Y$, and moreover whether such an R is greatest, which is the case in fuzzy relational equations. Here we remark that Sanchez [9] introduced the special kind of operators like ($\textcircled{\alpha}$, $\textcircled{\sigma}$) to compose fuzzy relations. He used these operators in the resolution of fuzzy relational equations. However it seems it is not possible to generalize these operators for the intuitionsitic fuzzy case.

5. Extension principle

Theorem 5.1 (Extension Principle). *Let X and Y be two universes. Let $A,B \in IFS(X)$ and f be a mapping from X to Y. Let $R \in IFR(A,B)$. Then $f(A)$ and $f(B)$ are $IFSs$ on Y and f_R an IFR from $f(A)$ to $f(B)$, which are defined as follows: $\forall y \in Y$*

$$\mu_{f(A)}(y) = \begin{cases} \sup\limits_{x\in f^{-1}(y)} \mu_A(x) & \text{if } f^{-1}(y) \ne \emptyset, \\ 0 & \text{otherwise,} \end{cases} \qquad \nu_{f(A)}(y) = \begin{cases} \inf\limits_{x\in f^{-1}(y)} \nu_A(x) & \text{if } f^{-1}(y) \ne \emptyset, \\ 1 & \text{otherwise} \end{cases}$$

Similarly $\mu_{f(B)}$ and $\nu_{f(B)}$ can be defined, and

$\forall y_1, y_2 \in Y$,

$$\mu_{f_R}(y_1,y_2) = \begin{cases} \sup\limits_{x_1\in f^{-1}(y_1),\, x_2\in f^{-1}(y_2)} \mu_R(x_1,x_2) & \text{if both } f^{-1}(y_1) \ne \emptyset, f^{-1}(y_2) \ne \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

$$\nu_{f_R}(y_1,y_2)=\begin{cases}\inf\limits_{x_1\in f^{-1}(y_1),x_2\in f^{-1}(y_2)}\nu_R(x_1,x_2) & \text{if both } f^{-1}(y_1)\neq\emptyset, f^{-1}(y_2)\neq\emptyset,\\ 0 & \text{otherwise}\end{cases}$$

Proof. It is clear that $f(A)$ and $f(B)$ are IFS on Y. We have to show that $f(R)$ is an IFR from $f(A)$ to $f(B)$. Let $y_1,y_2\in Y$. If either $f^{-1}(y_1)$ or $f^{-1}(y_2)$ is empty then $\mu_{f_R}(y_1,y_2)=0\leq\min\left(\mu_{f(A)}(y_1),\mu_{f(B)}(y_2)\right)$. Let both $f^{-1}(y_1)$ and $f^{-1}(y_2)$ be non-empty. Then for each $(x_1,x_2)\in f^{-1}(y_1)\times f^{-1}(y_2)$, we have $\mu_R(x_1,x_2)\leq\min\left(\mu_A(x_1),\mu_B(x_2)\right)$, taking sup on either side we get

$$\mu_{f_R}(y_1,y_2)=\sup_{x_1\in f^{-1}(y_1),x_2\in f^{-1}(y_2)}\mu_R(x_1,x_2)\leq\sup_{x_1\in f^{-1}(y_1),x_2\in f^{-1}(y_2)}\left(\min\left(\mu_A(x_1),\mu_B(x_2)\right)\right)$$

$$=\min\left(\sup_{x_1\in f^{-1}(y_1)}\mu_A(x_1),\sup_{x_2\in f^{-1}(y_2)}\mu_B(x_2)\right)=\min\left(\mu_{f(A)}(y_1),\mu_{f(B)}(y_2)\right).$$

Similarly it can be shown that $\nu_{f_R}(y_1,y_2)\geq\max\left(\nu_{f(A)}(y_1),\nu_{f(B)}(y_2)\right)$. Also it is clear that f_R satisfies the intuitionsitic condition. Hence the proof of the theorem.

6. Reflexivity, symmetry and transitivity

Definition 6.1. An IFR R on $A\in IFS(X)$ is reflexive of order (α,β) if $\forall x\in X$ $\mu_R(x,x)=\alpha$ and $\nu_R(x,x)=\beta$ such that $\mu_A(x)\neq 0$ and $\nu_A(x)\neq 1$.

Note 6.2. If R is reflexive of order (α,β), then $\mu_{R^{-1}}(x,x)=\mu_R(x,x)=\alpha$ and $\nu_{R^{-1}}(x,x)=\nu_R(x,x)=\beta$. So R^{-1} is also reflexive of order (α,β).

Theorem 6.3. *If R_1 and R_2 are IFR's on A of orders (α_1,β_1) and (α_2,β_2) respectively, then $R_1\cup R_2$, $R_1\cap R_2$, R_1+R_2, $R_1\cdot R_2$, $R_1\Cup R_2$, $R_1\Cap R_2$, $\Box R_1$, $\Diamond R_1$, $R_1 @ R_2$, $R_1\$R_2$, $R_1\# R_2$, $R_1\star R_2$ are reflexive of orders respectively,* $(\max(\alpha_1,\alpha_2),\min(\beta_1,\beta_2))$, $(\min(\alpha_1,\alpha_2),\max(\beta_1,\beta_2))$, $(\alpha_1+\alpha_2-\alpha_1\cdot\alpha_2)$, $(\beta_1\cdot\beta_2)$, $(\alpha_1\cdot\alpha_2,\beta_1+\beta_2-\beta_1\cdot\beta_2)$, $(\min(1,\alpha_1+\alpha_2),\max(0,\beta_1+\beta_2-1))$, $\max(0,\alpha_1+\alpha_2-1)$, $\min(1,\beta_1+\beta_2)$, $(\alpha_1,1-\alpha_1)$, $(1-\beta_1,\beta_1)$, $(\alpha_1+\alpha_2/2,\beta_1+\beta_2/2)$, $\left(\sqrt{\alpha_1\cdot\alpha_2},\sqrt{\beta_1\cdot\beta_2}\right)$, $(2\alpha_1\cdot\alpha_2/\alpha_1+\alpha_2,2\beta_1\cdot\beta_2/\beta_1+\beta_2)$, $\left(\alpha_1+\alpha_2/2(\alpha_1\cdot\alpha_2+1),\beta_1+\beta_2/2(\beta_1\cdot\beta_2+1)\right)$.

Proof. The proof follows from the definitions of the respective operations.

Definition 6.4. An IFR R on A is symmetric if and only if

$$\mu_R(x,y)=\mu_R(y,x) \text{ and } \nu_R(x,y)=\nu_R(y,x) \quad \forall x,y\in X.$$

Theorem 6.5. *R is symmetric implies R^{-1} is so.*

Proof. $\mu_{R^{-1}}(x,y)=\mu_R(y,x)=\mu_R(x,y)=\mu_{R^{-1}}(y,x)$ $\forall x,y\in X$. Similarly $\nu_{R^{-1}}(x,y)$ $=\nu_{R^{-1}}(y,x)$. Hence the proof. Also we have the following result.

Theorem 6.6. *If R_1 and R_2 are symmetric IFR's on A, then $R_1 * R_2$ is also symmetric on A, where $*$ stands for the operations $\cup$, $\cap$, $+$, $\cdot$, $\Cap$, $\Cup$, $@$, $\$$, $\#$, $\star$.*

Definition 6.7. An IFR R on $A\in IFS(X)$ is said to be transitive if $R^2(=R\circ R)\subseteq R$. We have the following results.

Theorem 6.8. *If R is a transitive relation on A then R^{-1} is so.*

Theorem 6.9. *If R_1, R_2 are transitive relations on A, then so is $R_1\cap R_2$.*

7. Conclusion

In this paper we have introduced the concept of intuitionsitic fuzzy relation over intuitionsitic fuzzy sets. Resolving intuitionsitic fuzzy relational equations will be our future research.

References

[1] Krassimir T. Atanassov, Intuitionistic fuzzy relations, in: L. Antonov, Ed., III International school "*Automation and Scientific Instrumentation*", Varna (Oct.), (1984), 56-57.

[2] Krassimir T. Atanassov, Intuitionsitic fuzzy sets, *Fuzzy Sets and Systems*, 20 (1986), 87-96.

[3] Krassimir T. Atanassov, More on intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 33(1) (1989), 37-46.

[4] Krassimir T. Atanassov, New operation defined over the intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 61 (1994), 137-142.

[5] Krassimir T. Atanassov, Two operators on intuitionistic fuzzy sets, *Comptes Rendus de 1' Academie Bulgare des sciences*, 41(5) (1988), 35-38.

[6] P. Burillo and H. Bustince, Intuitionistic fuzzy relations, Part-I, *Mathware and Soft Computing*, 2 (1995), 5-38.

[7] M. K. Chakraborty and Mili Das, Studies in fuzzy relations over fuzzy subsets, *Fuzzy Sets and Systems*, 9 (1983), 79-89.

[8] A. Kaufmann, Introduction to the theory of fuzzy subsets, Vol-I, Academic Press, New York, (1975).

[9] Elie Sanchez, Resolution of composite fuzzy relation equations, *Inform. And Control*, 30 (1) (1976), 38-48.

[10] L. A. Zadeh, Fuzzy sets, *Inform. And Control*, 8 (1965), 338-353.