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GENERALIZED IDEALS WITH A TRIANGULAR NORM

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ABSTRACT. The notion of generalized ideal is redefined with respect to a triangular norm for a completely distributive complete lattice with a greatest element and least element and the new mathematical object is termed as a T-g-ideal. We have furnished examples of T-g-ideals with different t-norms and shown that a T-g-ideal with respect to one t-norm may not be a T-g-ideal with respect to another t-norm. New T-g-ideals from old ones have been constructed through various poset operations like product of lattices, ordinal sum of lattices, dual of a lattice, interval of a lattice etc.

1. INTRODUCTION

In 1971 the concept of fuzzy subgroups was introduced by A. Rosenfeld [12] and subsequently it was redefined with the help of t-norms by Anthony and Sherwood [5] and it was named as t-fuzzy subgroups. Later many researchers have contributed to the study of t-fuzzy subgroups. Yuan and Wu [15] applied the concept of fuzzy set in lattice theory and introduced the notions of fuzzy sublattices and fuzzy ideals. Later on fuzzy lattices was extensively studied by N. Ajmal [1-4]. Ideals are of fundamental importance in algebra. Filters, the order dual of lattice ideals have a variety of applications in logic and topology. M. H. Burton et al. [6, 7] have generalized the notion of a filter and called the new mathematical object as a generalized filter. In [11] A. A. Ramadan et al. introduced *generalized ideal* (Definition 2.3) (which was defined on a power set) is the dual of a generalized filter. Taking motivation from [5], in this paper we define a generalized ideal for a completely distributive complete lattice (with a greatest element and a least element) with respect to a triangular norm (briefly a t-norm) and call it a *T-g-ideal*. We show with examples that T-g-ideals with different t-norms exist and a T-g-ideal w.r.t. one t-norm may not be a T-g-ideal w.r.t. a different t-norm. However, a T-g-ideal w.r.t. the minimum t-norm which is the strongest t-norm is a T-g-ideal w.r.t. all other t-norms.

We organize our paper as follows. Section 1 is introduction. In Section 2 we recall some relevant definitions, notation and results which will be needed in the sequel. In Section 3 we define a T-g-ideal and provide some examples. In Section 4 first we have recalled some classes of lattices and constructed T-g-ideals on them from known T-g-ideals.

2. PRELIMINARIES

Let $(P, \leq, \vee, \wedge, \hat{1}, \hat{0})$ be a bounded completely distributive complete lattice with partial order relation $\leq$, and the binary operations $\vee, \wedge$ respectively called join and meet are defined

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as $a \vee b := \sup\{a, b\}$ and $a \wedge b := \inf\{a, b\}$ $\quad (a, b \in P)$. The greatest element $\hat{1}$ (unique) has the property that $a \vee \hat{1} = \hat{1} = \hat{1} \vee a \quad \forall a \in P$, and the least element $\hat{0}$ (unique) has the property that $a \wedge \hat{0} = \hat{0} = \hat{0} \wedge a$. Recall that the complete distributivity of P means the distributive law $\vee_{k \in J}(a_k \wedge a) = (\vee_{k \in J} a_k) \wedge a$ holds. For $a \in P$, we say $b \in P$ is a *complement* of a if $a \vee b = \hat{1}$ and $a \wedge b = \hat{0}$. A lattice P is called a Boolean algebra if (i) P is distributive, i.e., $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \forall a, b, c \in P$, (ii) P has $\hat{1}, \hat{0}$, (iii) each element $a \in P$ has a (necessarily unique) complement $a' \in P$.

Throughout this paper we consider a lattice as a completely distributive complete lattice.

The concept of *t-norm* was introduced in [13] while working on probabilistic metric spaces. More details about t-norms and their applications can be found in the recent monographs [9] and [14]. As usual we write I to denote the closed unit interval $[0, 1]$. The definition of a t-norm is as follows:

Definition 2.1. A *triangular norm* (t-norm, for short) is a function $T: I \times I \to I$ such that $\forall x, y, z \in I$:

(1) $T(x, 1) = x$ (boundary condition);
(2) $T(x, y) = T(y, x)$ (commutativity);
(3) $x \leq y \Rightarrow T(x, z) \leq T(y, z)$ (monotonicity);
(4) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity).

It is clear that $T(x, 0) = T(0, x) = 0 \quad \forall x \in I$, i.e. 0 is the annihilator.

For a t-norm T an element $a \in]0, 1[$ is called a *zero divisor* of T if there exists some $b \in]0, 1[$ such that $T(a, b) = 0$.

The examples of t-norms which are frequently used in a fuzzy setting are the following:

(1) (Minimum norm) $T_M(x, y) = \min\{x, y\} \quad \forall x, y \in I$;
(2) (Product norm) $T_P(x, y) = xy \quad \forall x, y \in I$;
(3) (Lukasiewicz norm) $T_L(x, y) = \max\{x + y - 1, 0\} \quad \forall x, y \in I$.

Definition 2.2 ([13])**.** A t-norm T_1 is *stronger* than a t-norm T_2, if and only if $T_1(x, y) \geq T_2(x, y) \quad \forall x, y \in I$.

Lemma 2.1 ([13])**.** *T_M is the strongest of all t-norms.*

The function T is defined on $I \times I$. However the domain of the function can be generalized to I^n (see [10]). The commutativity and associativity of a t-norm T ensures its unique n-ary extension which will be denoted by T_n, i.e.,

$$T_n(x_1, x_2, \ldots, x_n) = T_n(x_i, T_{n-1}(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n))$$

for all $1 \leq i \leq n$, where $n \geq 2$, $T_2 = T$. Also the following may be noted.

(1) $T_n(x_1, x_2, \ldots, x_n) = 0$ if $x_j = 0$ for some $j, 1 \leq j \leq n$.
(2) If $x_j = 1$, then

$$T_n(x_1, x_2, \ldots, x_n) = T_{n-1}(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)).$$

Hence $T_n(x_1, x_2, \ldots, x_n) = x_i$ if $x_j = 1 \quad \forall j \neq i$.

(3) For α a permutation of $\{x_1, x_2, \ldots, x_n\}$, we have

$$T_n(x_1, x_2, \ldots, x_n) = T_n(\alpha(x_1, x_2, \ldots, x_n)).$$

(4) $T_n(x_1, x_2, \ldots, x_n) \leq T_n(x_1, x_2, \ldots, x_{j-1}, x_j^*, x_{j+1}, \ldots, x_n)$ if $x_j \leq x_j^*$ for some $j, 1 \leq j \leq n$.

(5) $T_n(x_1,\dots,x_{n-1},T_n(y_1,\dots,y_n)) = T_n(x_1,\dots,x_{n-2},T_n(x_{n-1},y_1,\dots,y_{n-1}),y_n)$.
(6) Let $a_i,b_i \in I \quad \forall i, 1 \le i \le n$ and $n \ge 2$. Then

$$T_n(T(a_1,b_1),T(a_2,b_2),\dots,T(a_n,b_n)) = T(T_n(a_1,a_2,\dots,a_n),T_n(b_1,b_2,\dots,b_n)).$$

However in this paper we will write T instead of T_n.

Definition 2.3 ([11])**.** Let X be a nonempty set. Let $P = \mathcal{P}(X)$ be the power set of X. A nonzero function $d\colon P \to I$ is called a *generalized-ideal* (g-ideal, for short) if the following conditions are satisfied:

(G1) $d(X) = 0$,
(G2) $A \subset B \Rightarrow d(A) \ge d(B), \quad \forall A,B \in P$,
(G3) $d(A \cup B) \ge d(A) \wedge d(B)) \quad \forall A,B \in P$.

3. T-G-IDEAL

Definition 3.1. Let P be a lattice. A nonzero function $\tau\colon P \to I$ is called a *T-generalized-ideal* (T-g-ideal for short) w.r.t. a t-norm T if the following conditions are satisfied:

(TG-1) $\tau(\hat{1}) = 0$,
(TG-2) $a \le b \Rightarrow \tau(a) \ge \tau(b), \quad \forall a,b \in P$,
(TG-3) $\tau(a \vee b) \ge T(\tau(a),\tau(b)) \quad \forall a,b \in P$.

Remark 3.1. By Lemma 2.1 and (TG-3), we can note that if τ is a T-g-ideal w.r.t. T_M then for any t-norm T, τ is also a T-g-ideal w.r.t. T. Also note that since τ is nonzero, condition (TG-2) suggests that $\tau(\hat{0}) = \sup_{a\in P}\tau(a) > 0$. When T is the minimum t-norm, conditions (TG-2) and (TG-3) become equivalent to the condition

$$\tau(a \vee b) = T(\tau(a),\tau(b)) \quad \forall a,b \in P. \tag{3.1}$$

But for any other t-norm (TG-2) and (TG-3) may not be equivalent to (3.1), which may be verified from the following example.

Example 3.1. Consider the Lukasiewicz t-norm, T_L:

$$T_L(x,y) = \max\{x+y-1,0\} \quad \forall a,b \in I.$$

Let $X = [n[:= \{1,2,\dots,n\}$ for some fixed $n \in \mathbb{N}$ (where $\mathbb{N}$ is the set of natural numbers). Let $P = \mathcal{P}(X)$, the power set of X, which is a lattice, with set inclusion as the order relation, and $\hat{1} := [n[$ and $\hat{0} := \phi$. Define $\tau : P \to I$ as follows:

$$\tau(A) = \begin{cases} 1 - \dfrac{\sum\limits_{i\in A} i}{m} & \text{if } A \in P, A \ne \phi, \text{ where } m = n(n+1)/2 \\ 1 & \text{if } A = \phi. \end{cases}$$

Then clearly $\tau(\hat{1}) = \tau([n[) = 1 - \dfrac{\sum\limits_{i\in[n[} i}{m} = 1 - \dfrac{m}{m} = 0$. Let $A \subseteq B \in P$, implies that $\tau(A) \ge \tau(B)$, since $\sum\limits_{i\in A} i \le \sum\limits_{i\in B} i \Rightarrow 1 - \dfrac{\sum\limits_{i\in A} i}{m} \ge 1 - \dfrac{\sum\limits_{i\in B} i}{m} \Rightarrow \tau(A) \ge \tau(B)$.

To prove condition (TG-3), let $A,B \in P$. Two cases may arise (i) $A \cap B = \phi$, or, (ii) $A \cap B \ne \phi$.

(i) If $A \cap B = \phi$, let $\sum_{i\in A} i = p, \sum_{i\in B} i = q$, then $\sum_{i\in A} \cup Bi = p+q$. Hence $\tau(A) = 1 - \frac{p}{m}$, $\tau(B) = 1 - \frac{q}{m}$ and $\tau(A \cup B) = 1 - \frac{p+q}{m}$. Now $T_L(\tau(A), \tau(B)) = \max\{1 - \frac{p}{m} + 1 - \frac{q}{m} - 1, 0\} = \max\{1 - \frac{p}{m} - \frac{q}{m}, 0\} = 1 - \frac{p+q}{m}$ (since $p + q \leq m$).

(ii) If $A \cap B \neq \phi$, let $\sum_{i\in A} i = p, \sum_{i\in B} i = q$, then $\sum_{i\in A\cup B} i = p + q - r$, where $r = \sum_{i\in A\cap B} i > 0$. Note that $p + q - r \leq m$. Hence $\tau(A) = 1 - \frac{p}{m}, \tau(B) = 1 - \frac{q}{m}$ and $\tau(A \cup B) = 1 - \frac{p+q-r}{m}$. Now $T_L(\tau(A), \tau(B)) = \max\{1 - \frac{p}{m} - \frac{q}{m}, 0\} = k(\text{say})$. If $1 - \frac{p+q}{m} < 0$, then $k = 0$ and (TG-3) is satisfied. If $1 - \frac{p+q}{m} \geq 0$, then

$$k = 1 - \frac{p+q}{m} < 1 - \frac{p+q-r}{m} \quad (\text{ as } r > 0) \tag{3.2}$$

Thus in all the cases condition (TG-3) is satisfied.

Hence τ is a T-g-ideal w.r.t. T_L.

Remark 3.2. Note (eqn. (3.2)) when $A \cap B \neq \phi$, $T_L(\tau(A), \tau(B)) < \tau(A \cup B)$.

The strict inequality is obtained for T_L-norm, which is not the case for a T-g-ideal w.r.t. minimum t-norm. This also suggests that the function τ defined above is not a T-g-ideal w.r.t. minimum t-norm.

In fact τ is also not a T-g-ideal w.r.t. product t-norm. In Theorem 3.1 we will show why this happened.

For a function $\tau : P \to I$ and $a \in P$, we use the following notation [11]

$$\langle\tau\rangle(a) := \bigvee_{a \leq b} \tau(b)$$

Definition 3.2. Let P be a lattice. A nonzero function $\tau : P \to I$ is called a *T-generalized-ideal base* (T-g-IB for short) w.r.t. a t-norm T if the following conditions are satisfied:

(TGB1) $\tau(\hat{1}) = 0$,
(TGB2) $\langle\tau\rangle(a \vee b) \geq T(\tau(a), \tau(b)) \quad \forall a, b \in P$.

Evidently, a T-g-ideal is a T-g-IB.

The following propositions are immediate.

Proposition 3.1. *If a function $\tau : P \to I$ is a T-g-IB, then $\langle\tau\rangle$ is a T-g-ideal.*

Proposition 3.2. *A T-g-IB $\tau : P \to I$ is a T-g-ideal if and only if $\tau = \langle\tau\rangle$.*

We furnish an example to show that a T-g-IB may not be a T-g-ideal.

Example 3.2. Let $X = [4[$. Let $P = P(X)$. We define a function $\tau : P \to I$ as follows:

$$\tau(A) = \begin{cases} 1/3 & \text{if } A = \phi \\ 1/4 & \text{if } A = \{1\} \\ 1/3 & \text{if } A = \{2\} \\ 1/6 & \text{if } A = \{3\} \\ 1/6 & \text{if } A = \{1,2\} \\ 1/5 & \text{if } A = \{1,3\} \\ 1/4 & \text{if } A = \{2,3\} \\ 0 & \text{otherwise.} \end{cases}$$

Here $\{3\} \subset \{1,3\}$ but $\tau(\{3\}) = 1/6 < 1/5 = \tau(\{1,3\})$. Hence τ is not a T-g-ideal w.r.t any t-norm. But it can be easily checked that τ is a T-g-IB w.r.t. T_L. Note that $T_L(x,y) = 0$, $\forall x, y \leq 0.5$.

But

$$\langle\tau\rangle(A) = \begin{cases} 1/4 & \text{if } A = \{3\} \\ \tau(A) & \text{otherwise} \end{cases}$$

is a T-g-ideal w.r.t. T_L.

But still then $\langle\tau\rangle$ is not a T-g-ideal w.r.t. minimum t-norm. Since

$$\min(\langle\tau\rangle(\{1\}), \langle\tau\rangle(\{2\})) = \min(1/4, 1/3) = 1/4.$$

But $\min\langle\tau\rangle(\{1\} \cup \{2\}) = \langle\tau\rangle(\{1,2\}) = 1/6 < 1/4$.

However, $\langle\tau\rangle$ is a T-g-ideal w.r.t. product t-norm.

Here we present a theorem on T-g-ideal w.r.t. product t-norm.

Let P be a lattice and let $x, y \in P$. We say x is *covered by* y (or y *covers* x) and denoted by $x \leftharpoondown y$ or $(y \rightharpoondown x)$, if $x < y$ and $x \leq z < y$ implies $z = x$. That means there can be no elements z of P with $x < z < y$. Let $\hat{0}$ be the least element of P. Then $a \in P$ is called an *atom* if $0 \leftharpoondown a$. The set of atoms of P is denoted by $\mathcal{A}(P)$. The lattice P is called *atomic* if given $a \neq 0$ in P, $\exists x \in \mathcal{A}(P)$ such that $x \leq a$. Every finite lattice is atomic. By contrast, it may happen that an infinite lattice has no atom at all. The chain of non-negative real numbers provides an example. Even a Boolean lattice may have no atoms (see [8].)

Theorem 3.1. *Let P be a finite Boolean algebra. Let $\tau : P \to I$ be a T-g-ideal w.r.t a t-norm with no zero divisors. Then there exists $a \in \mathcal{A}(P)$ such that $\tau(b) = 0$ $\forall b \geq a \in P$.*

Proof. If there exists $a \in \mathcal{A}(P)$ such that $\tau(a) = 0$ then by condition (TG-2), we have $\tau(b) = 0 \quad \forall b \geq a \in P$. Therefore we only have to prove the existence of $a \in \mathcal{A}(P)$ with $\tau(a) = 0$.

We note that a finite Boolean algebra is always a join of its atoms (finitely many). Let $a_1, a_2, \ldots, a_n$ be all the atoms of P. Then $\hat{1} = a_1 \vee a_2 \vee \ldots \vee a_n$. By TG-3, $\tau(a_1 \vee a_2 \vee \ldots \vee a_n) \geq T(\tau(a_1), \tau(a_2), \ldots \tau(a_n))$.

Since T is a t-norm with no zero divisors, and $0 = \tau(\hat{1}) = \tau(a_1 \vee a_2 \vee \ldots \vee a_n) = T(\tau(a_1), \tau(a_2), \ldots \tau(a_n))$, therefore, there exists at least one atom $a_j = 0$. ☐

The following example justifies the Theorem 3.1.

Example 3.3. Let $Y = [3[= \{1,2,3\}$. Let $P = \mathcal{P}(Y)$ be the power set of Y. Define $\tau{:}P \to I$ as follows:

$$\tau(A) = \begin{cases} 1/3 & \text{if } A = \phi \\ 1/4 & \text{if } A = \{1\} \\ 1/3 & \text{if } A = \{2\} \\ 1/4 & \text{if } A = \{3\} \\ 1/6 & \text{if } A = \{1,2\} \\ 1/5 & \text{if } A = \{1,3\} \\ 1/4 & \text{if } A = \{2,3\} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\tau(a) \neq 0$ for all $\mathcal{A}(P) = \{\{1\},\{2\},\{3\}\}$. By Theorem 3.1 τ can not be a T-g-ideal w.r.t. the product norm T_P.

Now reconstruct the above example as follows:

Let $X = Y \cup \{4\} = \{1,2,3,4\}$ and define $\psi : \mathcal{P}(X) \to I$ by

$$\psi(A) = \begin{cases} \tau(A) & \text{if } A \subsetneqq Y \\ 1/6 & \text{if } A = Y \\ 0 & \text{otherwise.} \end{cases}$$

It can be easily verified that ψ is a T-g-ideal.

4. CONSTRUCTION OF T-G-IDEALS

Suppose P and Q are two lattices with partial orders $\leq_P$ and $\leq_Q$ respectively. Similarly we will use the notations $\vee_P, \wedge_P, \hat{1}_P, \hat{0}_P$ etc. for join, meet, greatest element, least element respectively, with subscript as the corresponding lattice. When there is no confusion we might omit the subscript at places. In general $\{P_i\}_{\{i=1,2,\dots,n\}}$ be lattices with partial order relations $\leq_{P_i}$.

Here we consider the following lattices (see [8]) to construct T-g-ideals on them.

(1) The direct product $P \times Q$ is a lattice on the product set with the order relation
$$(x_1, y_1) \leq (x_2, y_2) \text{ in } P \times Q \text{ iff } x_1 \leq_P x_2 \text{ and } y_1 \leq_Q y_2$$
also $(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee_P x_2, y_1 \vee_Q y_2)$ similarly the meet operation can be defined. Here note that $\hat{1}_{P\times Q} := (\hat{1}_P, \hat{1}_Q)$, and $\hat{0}_{P\times Q} := (\hat{0}_P, \hat{0}_Q)$.

(2) Similarly the direct product $P := \prod_{i=1}^{n} P_i$ of n lattices $\{P_i\}_{\{i=1,2,\dots,n\}}$ can be defined.

(3) The ordinal sum $P \oplus Q$ is a lattice on the disjoint union of P and Q with the order relation $x \leq y$ in $P \oplus Q$ iff one of (a) $x, y \in P$ and $x \leq_P y$, or (b) $x, y \in Q$ and $x \leq_Q y$, or(c) $x \in P$ and $y \in Q$ holds good.

Here $\hat{1}_{P\oplus Q} = \hat{1}_Q$, $\hat{0}_{P\oplus Q} = \hat{0}_P$.

(4) Similarly the ordinal sum $\oplus_{i=1}^{n} P_i := P_1 \oplus P_2 \oplus \cdots \oplus P_n$ of n lattices $\{P_i\}_{i=1,2,\dots,n\}}$ is a lattice on the disjoint union of P_i's with order relation as follows: $x \leq y$ in $\oplus_{i=1}^{n} P_i$ iff (a) $x, y \in P_i$ for some i, and $x \leq_{P_i} y$ or (b) $x \in P_i, y \in P_j$ and $i < j$. Also $\hat{1}_{\oplus_{i=1}^{n} P_i} := \hat{1}_{P_n}, \hat{0}_{\oplus_{i=1}^{n} P_i} := \hat{0}_{P_1}$.

(5) The cardinal power Q^P is the lattice on the set of order preserving maps $f : P \to Q$ (i.e. $\forall x_1, x_2 \in P, x_1 \leq_P x_2 \Rightarrow f(x_1) \leq_Q f(x_2)$), with partial order relation:
$$f \leq g \text{ in } Q^P \text{ iff } f(x) \leq_Q g(x) \quad \forall x \in P.$$
The greatest element in Q^P is the map $\hat{1}_{Q^P}\colon P \to Q$ defined by $\hat{1}_{Q^P}(x) = \hat{1}_Q \quad \forall x \in P$. Similarly the least element in Q^P is the map $\hat{0}_{Q^P}\colon P \to Q$ defined by $\hat{0}_{Q^P}(x) = \hat{0}_Q$, $\forall x \in P$.

(6) The *dual lattice* P^* of P is a lattice on the same set P with order relation $x \leq_{P^*} y$ iff $y \leq_P x$. Thus $\hat{1}_{P^*} := \hat{0}_P$, $\hat{0}_{P^*} := \hat{1}_P$, $x \vee^* y := x \wedge y$ and $x \wedge^* y := x \vee y$.

If x' is the complement of x in P, i.e. $x \vee x' = \hat{1}_P$ and $x \wedge x' = \hat{0}_P$ then x' is also the complement of x in P^*, with $x \vee^* x' = \hat{1}_{P^*} = \hat{0}_P = x \wedge x'$ and $x \wedge^* x' = \hat{0}_{P^*} = \hat{1}_P = x \vee x'$.

(7) Let P be a lattice. Let $x \in P$. Then both the intervals $[\hat{0}, x]$ and $[x, \hat{1}]$ are sublattices of P.

Theorem 4.1. *Let $P := \prod_{i=1}^{n} P_i$ be the direct product of n lattices $\{P_i\}_{\{i=1,2,\dots,n\}}$. Let $\tau_i : P_i \to I$ be a T-g-ideal for each $i = 1, 2, \dots, n$ w.r.t. the t-norm T. Then the function $\tau : P \to I$ defined by*

$$\tau(\underline{a}) := T(\tau_1(a_1), \tau_2(a_2), \dots, \tau_n(a_n)) \quad \forall \underline{a} := (a_1, a_2, \dots, a_n) \in P$$

is a T-g-ideal on P if $\tau(\hat{0}_P) = T(\hat{0}_{P_1}, \hat{0}_{P_2}, \ldots, \hat{0}_{P_n}) \neq 0$.

Proof. Since $\tau(\hat{0}_P) \neq 0$, τ is a nonzero function. We will check the three conditions for τ to be a T-g-ideal.

(1) $\tau(\hat{1}_P) = T(\tau(\hat{1}_{P_1}), \tau(\hat{1}_{P_2}), \ldots, \tau(\hat{1}_{P_n})) = T(0, 0, \ldots, 0) = 0$.

(2) Let $\underline{a}(= (a_1, a_2, \ldots, a_n)) \leq \underline{b}(= (b_1, b_2, \ldots, b_n)) \in P$. Then $a_i \leq_{P_i} b_i \forall i = 1, 2, \ldots, n$. Now

$$\tau(\underline{a}) = T(\tau_1(a_1), \tau_2(a_2), \ldots, \tau_n(a_n)) \geq T(\tau_1(b_1), \tau_2(b_2), \ldots, \tau_n(b_n))$$

($\because \tau_i(a_i) \geq_{P_i} \tau_i(b_i)$ for each i, and also T is monotonic in each of the coordinates) = $\tau(\underline{b})$.

(3) Consider $\underline{a}, \underline{b} \in P$.

$$\begin{aligned}\tau(\underline{a} \vee \underline{b}) &= \tau(a_1 \vee_{P_1} b_1, \ldots, a_i \vee_{P_i} b_i, \ldots, a_n \vee_{P_n} b_n)\\ &= T(\tau(a_1 \vee_{P_1} b_1), \ldots, \tau(a_i \vee_{P_i} b_i), \ldots, \tau(a_n \vee_{P_n} b_n))\\ &\geq T(T(\tau_1(a_1), \tau_1(b_1)), \ldots, T(\tau_i(a_i), \tau_i(b_i)), \ldots, T(\tau_n(a_n), \tau_1(b_n))\\ &= T(T(\tau_1(a_1), \ldots, \tau_n(a_n)), T(\tau_1(a_1), \ldots, \tau_n(a_n))\tau_1(b_1))).\end{aligned}$$

Therefore τ is a T-g-ideal on P. ☐

Conversely let $P := \prod_{i=1}^{n} P_i$ be the direct product of n lattices $\{P_i\}_{\{i=1,2,\ldots,n\}}$. Let $\tau: P \to I$ be a T-g-ideal w.r.t. a t-norm T. Then can we derive some T-g-ideal on each P_i?

The answer is affirmative which is our next theorem.

Theorem 4.2. *Let $P := \prod_{i=1}^{n} P_i$ be the direct product of n lattices $\{P_i\}_{\{i=1,2,\ldots,n\}}$. Let $\tau : P \to I$ be a T-g-ideal on P w.r.t. a t-norm T. Assume that $\tau(\hat{0}_{P_1}, \hat{0}_{P_2}, \ldots, \hat{1}_{P_i}, \ldots, \hat{0}_{P_n}) = 0$, $\forall i \in [n[$. For each $i \in [n[$, define $\tau_i : P_i \to I$ by $\tau_i(a_i) = \tau(\hat{0}_{P_1}, \hat{0}_{P_2}, \ldots, a_i, \ldots, \hat{0}_{P_n})$ for each $a_i \in P_i$. Then τ_i defined this way is a T-g-ideal.*

Proof. Since τ is a nonzero function, $\tau(\hat{0}_P) > 0$ and hence $\tau_i(\hat{0}_{P_i}) = \tau(\hat{0}_{P_1}, \ldots, \hat{0}_{P_i}, \ldots, \hat{0}_{P_n}) = \tau(\hat{0}_P) > 0$.

(1) Now $\tau_i(\hat{1}_{P_i}) = \tau(\hat{0}_{P_1}, \ldots, \hat{1}_{P_i}, \ldots, \hat{0}_{P_n}) = 0$.

(2) Let $a_i \leq_{P_i} b_i$. Then $\tau_i(a_i) = \tau(\hat{0}_{P_1}, \ldots, a_i, \ldots, \hat{0}_{P_n}) \geq \tau(\hat{0}_{P_1}, \ldots, b_i, \ldots, \hat{0}_{P_n}) = \tau_i(b_i)$.

(3) Let $a_i, b_i \in P_i$.

$$\begin{aligned}\tau_i(a_i \vee b_i) &= \tau(\hat{0}_{P_1}, \ldots, a_i \vee b_i, \ldots, \hat{0}_{P_n})\\ &\geq T[\tau(\hat{0}_{P_1}, \ldots, a_i, \ldots, \hat{0}_{P_n}), \tau(\hat{0}_{P_1}, \ldots, b_i, \ldots, \hat{0}_{P_n})]\\ &= T(\tau_i(a_i), \tau_i(b_i)).\end{aligned}$$

Hence τ_i is a T-g-ideal on P_i for each i. ☐

Definition 4.1. Let P be a complete lattice. A function $\tau : P \to I$ is called a T-g-*preideal* w.r.t. a t-norm T if

(TGP1) $\tau(\hat{1}) > 0$,

(TGP2) $a \leq b \Rightarrow \tau(a) \geq \tau(b), \quad \forall a, b \in P$,

(TGP3) $\tau(a \vee b) \geq T(\tau(a), \tau(b)) \quad \forall a, b \in P$.

Now we will consider the ordinal sum $P \oplus Q$ of two complete lattices P and Q.

Theorem 4.3. *Let $P \oplus Q$ be the ordinal sum of two complete lattices P and Q with order relations as defined before. Let $\tau : P \to I$ be a T-g-preideal w.r.t. a t-norm T, and $\psi : Q \to I$ be a T-g-ideal w.r.t. the t-norm T such that $\tau(\hat{1}) \geq \psi(\hat{0})$. Then the function*

$$\tau \oplus \psi : P \oplus Q \to I$$

defined by

$$(\tau \oplus \psi)(x) = \begin{cases} \tau(x) & \text{if } x \in P \\ \psi(x) & \text{if } x \in Q \end{cases}$$

is a T-g-ideal on $P \oplus Q$ w.r.t. the t-norm T.

Proof. (1) As the maximal element in $P \oplus Q$ is $\hat{1}_Q$, $(\tau \oplus \psi)(\hat{1}_Q) = \psi(\hat{1}_Q) = 0$.

(2) Let $x \leq_{P\oplus Q} y$, then three cases may arise. (a) $x, y \in P$ and $x \leq_P y$, but then $(\tau \oplus \psi)(x) = \tau(x) \geq \tau(y) = (\tau \oplus \psi)(y)$. (b) If $x, y \in Q$ and $x \leq_Q y$, then similar to above. (c) If $x \in P, y \in Q$, then $(\tau \oplus \psi)(x) = \tau(x) \geq \tau(\hat{1}) \geq \psi(\hat{0}) \geq \psi(y) = (\tau \oplus \psi)(y)$.

(3) Let $x, y \in P \oplus Q$. Here also we have to do for all the three cases. When $x, y \in P$ or $x, y \in Q$ then $\tau \oplus \psi$ coincides with τ or ψ respectively. Hence (TG-3) is satisfied. If $x \in P, y \in Q$, then $x \vee y = y$. Therefore, $(\tau \oplus \psi)(x \vee y) = (\tau \oplus \psi)(y) = \psi(y)$. Now $T((\tau \oplus \psi)(x), (\tau \oplus \psi)(y)) = T(\tau(x), \psi(y)) \geq T(1, \psi(y)) = \psi(y) = (\tau \oplus \psi)(x \vee y)$.

Therefore, $\tau \oplus \psi$ is a T-g-ideal. □

The above theorem can be extended to the ordinal sum of finitely many complete lattices.

Theorem 4.4. *Let $P := \bigoplus_{i=1}^{n} P_i$ be the ordinal sum of n lattices $\{P_i\}_{\{i=1,2,\ldots,n\}}$. Let $\tau_i : P_i \to I$ be a T-g-preideal for each $i = 1, 2, \ldots, n-1$ w.r.t. a fixed t-norm T and $\tau_n : P_n \to I$ be a T-g-ideal w.r.t. the t-norm T such that $\tau_1(\hat{1}) \geq \tau_2(\hat{0}) \geq \tau_2(\hat{1}) \geq \cdots \geq \tau_i(\hat{0}) \geq \tau_i(\hat{1}) \geq \cdots \geq \tau_n(\hat{0})$. Then the function $\tau := (\tau_1 \oplus \tau_2 \oplus \cdots \oplus \tau_n) : P \to I$ defined by*

$$\tau(x) = \tau_i(x) \text{ if } x \in P_i, i = 1, 2, \ldots, n$$

Proof. The proof is similar to the Theorem 4.3. □

Before going to the next theorem on ordinal sum of lattices here we write the definition of ordinal sum of a family of t-norms.

Definition 4.2. Let $(T_\alpha)_{\alpha\in\Lambda}$ be a family of t-norms and $(]a_\alpha, e_\alpha[)_{\alpha\in\Lambda}$ be a family of non-empty pairwise disjoint open subintervals of I. Then the t-norm $T : I^2 \to I$ defined by

$$T(x,y) = \begin{cases} a_\alpha + (e_\alpha - a_\alpha) \cdot T_\alpha\left(\dfrac{x - a_\alpha}{e_\alpha - a_\alpha}, \dfrac{y - a_\alpha}{e_\alpha - a_\alpha}\right) & \text{if } (x,y) \in [a_\alpha, e_\alpha]^2, \\ \min(x,y) & \text{otherwise} \end{cases} \tag{4.3}$$

is called the *ordinal sum* of the summands $\langle a_\alpha, e_\alpha, T_\alpha\rangle, \alpha \in \Lambda$, and is denoted by $T = (\langle a_\alpha, e_\alpha, T_\alpha\rangle)_{\alpha\in\Lambda}$.

For a proof that T defined in (4.3) is a t-norm see ([9] Theorem 3.43).

Theorem 4.5. *Let $P := \bigoplus_{i=1}^{n} P_i$ be the ordinal sum of n complete lattices P_i with order relations as defined before. Let $\tau_i : P_i \to I$ be a T-g-ideal w.r.t. t-norm T_i, for each $i =$*

$1, 2, \ldots, n$. *Let* $0 = c_0 < c_1 < \cdots < c_{n-1} < c_n = 1$ *and consider the ordinal sum of the summands* $\langle c_{n-i}, c_{n-i+1}, T_i\rangle, i = 1, 2, \ldots, n$, *denoted by* $T = (\langle c_{n-i}, c_{n-i+1}, T_i\rangle)_{i\in[n[}$. *Then the function*

$$\phi : P \to I$$

defined by

$$\phi(x) = c_{n-i} + (c_{n-i+1} - c_{n-i})\tau_i(x) \quad \text{for } x \in P_i$$

is a T-g-ideal on P *w.r.t. the t-norm* T.

Proof. Clearly ϕ is nonzero.

(1) $\phi(\hat{1}_P) = \phi(\hat{1}_{P_n}) = c_0 + (c_1 - c_0)\tau_n(\hat{1}_{P_n}) = 0$, as $c_0 = 0, \tau_n(\hat{1}_{P_n}) = 0$.

(2) Let $x \le_P y$, then two cases may arise.

(a) When $x, y \in P_i$ and $x \le_{P_i} y$, so $\tau_i(x) \ge \tau_i(y)$ as τ_i is a T-g-ideal and we have $\phi(x) = c_{n-i} + (c_{n-i+1} - c_{n-i})\tau_i(x) \ge c_{n-i} + (c_{n-i+1} - c_{n-i})\tau_i(y) = \phi(y)$.

(b) When $x \in P_i$, $y \in P_j$, $i < j$, then $n - i \ge n - j + 1$ and so $c_{n-i} \ge c_{n-j+1}$. Now

$$\begin{aligned}\phi(x) &= c_{n-i} + (c_{n-i+1} - c_{n-i})\tau_i(x) \ge c_{n-j+1}\\ &= c_{n-j} + (c_{n-j+1} - c_{n-j}) \ge c_{n-j} + (c_{n-j+1} - c_{n-j})\tau_j(y) = \phi(y).\end{aligned}$$

Hence TG-2 is satisfied.

(3) Let $x, y \in P$. Here also we have to do for both the cases.

(a) When $x, y \in P_i$, then $x \vee y \in P_i$ and so $\phi(x \vee y) = c_{n-i} + (c_{n-i+1} - c_{n-i})\tau_i(x \vee y) \ge c_{n-i} + (c_{n-i+1} - c_{n-i})T_i(\tau(x), \tau(y))$. But by applying (4.3), we get

$$\begin{aligned}T(\phi(x), \phi(y)) &= T(c_{n-i} + (c_{n-i+1} - c_{n-i})\tau_i(x), c_{n-i} + (c_{n-i+1} - c_{n-i})\tau_i(y))\\ &= c_{n-i} + (c_{n-i+1} - c_{n-i})T_i(\tau(x), \tau(y)).\end{aligned}$$

Hence $\phi(x \vee y) \ge T(\phi(x), \phi(y))$.

(b) When $x \in P_i$, $y \in P_j$, $i < j$, then $x \vee y = y \in P_j$ and $c_{n-i} \ge c_{n-j}$. Also $c_{n-i} + (c_{n-i+1} - c_{n-i})\tau_i(x) \ge c_{n-j} + (c_{n-j+1} - c_{n-j})\tau_j(y)$.
Now $\phi(x \vee y) = c_{n-j} + (c_{n-j+1} - c_{n-j})\tau_j(x \vee y) = c_{n-i} + (c_{n-j+1} - c_{n-j})\tau_j(y)$, but

$$\begin{aligned}T(\phi(x), \phi(y)) &= T(c_{n-i} + (c_{n-i+1} - c_{n-i})\tau_i(x), c_{n-j} + (c_{n-j+1} - c_{n-j})\tau_j(y))\\ &= \min(c_{n-i} + (c_{n-i+1} - c_{n-i})\tau_i(x), c_{n-j} + (c_{n-j+1} - c_{n-j})\tau_j(y))\\ &= c_{n-j} + (c_{n-j+1} - c_{n-j})\tau_j(y).\end{aligned}$$

Hence $\phi(x \vee y) \ge T(\phi(x), \phi(y))$.

Hence (TG-3) is satisfied.

Therefore, ϕ is a T-g-ideal. □

Notation: Let $S = \{x_1, x_2, \ldots, x_n\}$ be a finite set. Let $f\colon S \to I$, then we use the notation $\underset{(x,S)}{T}(f(x)) := T(f(x_1), f(x_2), \ldots, f(x_n))$ in the following theorem.

Theorem 4.6. *Let* Q^P *be the cardinal power of lattices as defined before. Let* P *and* Q *be finite. Let* $\psi : Q \to I$ *be a T-g-ideal w.r.t. the t-norm* T *and* $\underset{(x,P)}{T}(\psi(\hat{0}_{Q^P}(x))) > 0$. *Then the function* $\psi^* : Q^P \to I$ *defined by* $\psi^*(f) = \underset{(x,P)}{T}(\psi(f(x)))$ *is a T-g-ideal.*

Proof. From the construction of ψ^*, we have $\psi^*(\hat{0}_{Q^P}) = \underset{(x,P)}{T}(\psi(\hat{0}_{Q^P}(x))) > 0$ (assumption on ψ). Hence ψ^* is a nonzero function.

(1) $\psi^*(\hat{1}_{Q^P}) = \underset{(x,P)}{T}(\psi(\hat{1}_{Q^P}(x))) = 0 \quad (\because \psi(\hat{1}_{Q^P}(y)) = \psi(\hat{1}_Q) = 0 \quad \forall y \in P$ and 0 is an annihilator for T).
(2) Let $f \leq_{Q^P} g$ where f, g are order preserving maps from P to Q. $f \leq_{Q^P} g \Rightarrow f(x) \leq g(x) \forall (x, P) \Rightarrow \psi(f(x)) \geq \psi(g(x)) \forall x \in P$ using this result and the monotonicity of T in each coordinate, we have $\psi^*(f) = \underset{(x,P)}{T}(\psi(f(x))) \geq \underset{(x,P)}{T}(\psi(g(x))) = \psi^*(g)$.
(3) Let $f, g \in Q^P$, then

$$\begin{aligned}\psi^*(f \vee g) &= \underset{(x,P)}{T}(\psi((f \vee g)(x))) = \underset{(x,P)}{T}(\psi(f(x) \vee g(x)))\\ &\geq \underset{(x,P)}{T}(T(\psi(f(x)), \psi(g(x)))) = T(\underset{(x,P)}{T}(\psi(f(x))), \underset{(x,P)}{T}(\psi(g(x))))\\ &= T(\psi^*(f), \psi^*(g)).\end{aligned}$$

Thus ψ^* is a T-g-ideal. □

Theorem 4.7. *Let P be a Boolean algebra. Let $\tau : P \to I$ be a T-g-ideal w.r.t. a t-norm T. Then the map $\tau^* : P^* \to I$ (P^* is the dual Boolean algebra of P), defined by $\tau^*(x) = \tau(x')$ where, x' is the complement of $x \in P$.*

Proof. Since τ is nonzero, $\tau^*(x)$ is also nonzero.

(1) Note that $\hat{1}_{P^*} = \hat{0}_P$. Now $\tau^*(\hat{1}_{P^*}) = \tau^*(\hat{0}_P) = \tau(\hat{1}_P) = 0$.
(2) Let $x \leq_{P^*} y \Rightarrow y \leq_P x \Rightarrow x' \leq_P y' \Rightarrow \tau(x') \geq \tau(y')$. So $\tau^*(x) = \tau(x') \geq \tau(y') = \tau^*(y)$.
(3) Finally let $x, y \in P^*$, $\tau^*(x \vee^* y) = \tau^*(x \wedge y) = \tau((x \wedge y)') = \tau(x' \vee y') \geq T(\tau(x'), \tau(y')) = T(\tau^*(x), \tau^*(y))$.

Thus τ^* is a T-g-ideal. □

Theorem 4.8. *Let P be a Boolean algebra. Let $\tau : P \to I$ be a T-g-ideal w.r.t. a t-norm T. Let $x \in P$, then the map $\tau|_x := \tau|_{[\hat{0},x]} : [\hat{0}, x] \to I$ defined by $\tau|_x(y) = \tau(y) \quad \forall y \in [\hat{0}, x]$, is*

(i) *a T-g-ideal w.r.t. the t-norm T if $\tau(x) = 0$,*
(ii) *otherwise a T-g-preideal w.r.t. the t-norm T.*

Proof. The proof is easy and we omit it. □

Theorem 4.9. *Let P be a Boolean algebra. Let $\tau : P \to I$ be a T-g-ideal w.r.t. a t-norm T. Let $x \in P$, then the map $\tau|^x := \tau|_{[x,\hat{1}]} : [x, \hat{1}] \to I$ defined by $\tau|^x(y) = \tau(y) \quad \forall y \in [x, \hat{1}]$, is a T-g-ideal w.r.t. the t-norm T if $\tau(x) > 0$.*

Proof. The proof is easy and we omit it. □

5. CONCLUSION

This paper deals with T-g-ideals for a completely distributive complete lattice with respect to a triangular norm. Examples of T-g-ideal with respect to different t-norms have been constructed. Existence of an atom with T-g-ideal value as zero is proved with respect to a t-norm with no zero divisors. Also T-g-ideals have been constructed on product lattices, ordinal sum of lattices, dual lattice, interval lattices etc. This research work can be extended to T-g-prime ideals, and study of their images, pre-images etc. Lattice ideal theory has application in domains and information systems. Concept analysis [8] provides a powerful technique in information science for classifying and analyzing compressed sets of data. It builds a partially ordered set which reveals inherent hierarchical structures and natural sub grouping and dependencies among objects and attributes. The concept of T-g-ideal can be used in formal concept analysis, since a context with a suitable partial order becomes a complete lattice.

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