Available online at www.sciencedirect.com
ScienceDirect
European Journal of Operational Research 185 (2008) 47–62
EUROPEAN JOURNAL OF OPERATIONAL RESEARCH
www.elsevier.com/locate/ejor

Continuous Optimization

Convex fuzzy mapping with differentiability and its application in fuzzy optimization

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Received 8 July 2005; accepted 29 December 2006
Available online 12 February 2007

Abstract

The concepts of differentiability, convexity, generalized convexity and minimization of a fuzzy mapping are known in the literature. The purpose of this present paper is to extend and generalize these concepts to fuzzy mappings of several variables using Buckley–Feuring approach for fuzzy differentiation and derive Karush–Kuhn–Tucker condition for the constrained fuzzy minimization problem.

Keywords: Fuzzy numbers; Differentiation; Convex analysis; KKT condition

1. Introduction

The theory of fuzzy differential calculus has been discussed by many researchers like Goetschel–Voxman [8], Seikkala [12], Puri–Ralescu [11], Dubois–Prade [5,6], and Friedman–Ming–Kandel [7]. A comparison of these various definitions has been discussed by Buckley–Feuring [3]. Goetschel–Voxman, Puri–Ralescu, and Friedman–Ming–Kandel have used non-standard fuzzy subtraction to define derivative of a fuzzy mapping. Buckley–Feuring [2,3] also have defined the derivative of a fuzzy mapping using left and right-hand functions of its α-level sets and established sufficient conditions for the existence of fuzzy derivative. Existence of Buckley–Feuring derivative implies the existence of above derivatives. Hence in this paper we accept the concept of differentiability of fuzzy mapping due to Buckley–Feuring [2,3].

Nanda and Kar [10] introduced the concept convexity for fuzzy mapping and proved that a fuzzy mapping is convex if and only if its epigraph is a convex set. Yan–Xu [16] have discussed convexity, quasiconvexity of fuzzy mappings by considering the concept of ordering due to Goetschel–Voxman [8]. Syau in [13] has proved

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[1] Research work is supported by AICTE, India, and G. H. Patel College of Engg. and Tech., V. V. Nagar 388 120, Gujarat, India, and IIT Kharagpur.
[2] The second author is thankful to ISIRD, IIT Kharagpur for financial support to prepare this paper.

0377-2217/$ - see front matter
doi:10.1016/j.ejor.2006.12.053

some results on convex fuzzy mapping and in [14], introduced the concept of differentiability, generalized convexity such as pseudoconvexity and invexity for fuzzy mappings of several variables. His approach is parallel to Goetschel–Voxman approach for fuzzy mapping of single variable in which the set of fuzzy numbers are embedded in a topological vector space.

Nanda–Kar [10] have defined the concept of quasiconvex fuzzy mapping and have discussed some applications to optimization. However, since the set of fuzzy numbers is a partially ordered set, it might happen that two fuzzy numbers may not be comparable (see Definition 3.10). Thus, in such case one is not sure what is the maximum or minimum of two fuzzy numbers (that is, when they are not comparable). So to overcome this difficulty, Syau [13] has taken a different approach and defined the supremum and infimum for a pair of fuzzy numbers (see Lemma 2.6). Accordingly he modified the definition of a quasiconvex fuzzy mapping of Nanda–Kar [10]. However, when we deal with the problem of minimization of a fuzzy mapping by considering the infimum defined by Syau [13], we might get a unique infimum of the fuzzy mapping (provided it is bounded). But in the process we might loose the solution, or in other words, there might not be any point (solution) (in the domain) at which the fuzzy mapping will have the value equal to infimum (see Example 3.10). So to overcome this difficulty we have modified the definition of quasiconvex fuzzy mapping in Section 4, which is different from Syau [13] as well as Nanda and Kar [10].

Section 3 deals with the minimization of a fuzzy mapping and also we introduce the gradient of a fuzzy function, directional derivative of a fuzzy function and establish the condition for a local minimum of a fuzzy differentiable function. The concept of convex fuzzy mapping, generalized convex fuzzy mapping such as quasiconvexity, strict quasiconvexity, strong quasiconvexity, pseudoconvexity using differentiability concept is introduced in Section 4. Section 5 deals with the Karush–Kuhn–Tucker optimality conditions for the constrained fuzzy minimization problem.

2. Preliminaries

We first quote some preliminary notations, definitions and results which will be needed in the sequel.

Definition 2.1. Let $\mathbb{R}$ denote the set of all real numbers. A fuzzy number is a mapping $\tilde{u} : \mathbb{R} \to [0,1]$ with the following properties:

1. $\tilde{u}$ is normal, that is, the core of $\tilde{u} = \text{core}(\tilde{u}) = \{x \in \mathbb{R} : \tilde{u}(x) = 1\}$ is not empty,
2. $\tilde{u}$ is upper semi-continuous,
3. $\tilde{u}$ is convex, that is,
$$\tilde{u}(\lambda x + (1 - \lambda y) \geqslant \min\{\tilde{u}(x), \tilde{u}(y)\}$$
for all $x, y \in \mathbb{R}, \lambda \in [0,1]$,
4. the support of $\tilde{u}$, supp $\tilde{u} = \{x \in \mathbb{R} : \tilde{u}(x) > 0\}$ and its closure cl(supp $\tilde{u}$) is compact.

Let $\mathscr{F}$ be the set of all fuzzy numbers on $\mathbb{R}$. The α-level set of a fuzzy number $\tilde{u} \in \mathscr{F}$, $0 \leqslant \alpha \leqslant 1$, denoted by $\tilde{u}[\alpha]$, is defined as

$$\tilde{u}[\alpha] = \begin{cases} \{x \in \mathbb{R} : \tilde{u}(x) \geqslant \alpha\} & \text{if } 0 < \alpha \leqslant 1, \\ cl(\text{supp}\,\tilde{u}) & \text{if } \alpha = 0. \end{cases}$$

It is clear that the α-level set of a fuzzy number is a closed and bounded interval $[u_\star(\alpha), u^\star(\alpha)]$, where $u_*(\alpha)$ denotes the left-hand end point of $\tilde{u}[\alpha]$ and $u^*(\alpha)$ denotes the right-hand endpoint of $\tilde{u}[\alpha]$.

Also any $m \in \mathbb{R}$ can be regarded as a fuzzy number $\tilde{m}$ defined by

$$\tilde{m}(t) = \begin{cases} 1 & \text{if } t = m, \\ 0 & \text{if } t \neq m. \end{cases}$$

In particular, the fuzzy number $\tilde{0}$ is defined as $\tilde{0}(t) = 1$ if $t = 0$, and $\tilde{0}(t) = 0$ otherwise.

Thus a fuzzy number $\tilde{u}$ can be identified by a parameterized triples

$$\{(u_*(\alpha), u^*(\alpha), \alpha) : 0 \leqslant \alpha \leqslant 1\}.$$

This leads to the following characterization of a fuzzy number in terms of the two "end point" functions $u_*(\alpha)$ and $u^*(\alpha)$.

Lemma 2.2 (Goetschel and Voxman ([8, Theorem 1.1])). *Assume that $I = [0,1]$, and $u_* : I \to \mathbb{R}$ and $u^* : I \to \mathbb{R}$ satisfy the conditions*:

1. $u_* : I \to \mathbb{R}$ *is a bounded increasing function,*
2. $u^* : I \to \mathbb{R}$ *is a bounded decreasing function,*
3. $u_*(1) \leqslant u^*(1)$,
4. *for* $0 < k \leqslant 1, \lim_{\alpha\to k^-} u_*(\alpha) = u_*(k)$ *and* $\lim_{\alpha\to k^-} u^*(\alpha) = u^*(k)$,
5. $\lim_{\alpha\to 0^+} u_*(\alpha) = u_*(0)$ *and* $\lim_{\alpha\to 0^+} u^*(\alpha) = u^*(0)$.

Then $\tilde{u} : \mathbb{R} \to I$ defined by

$$\tilde{u}(x) = \sup\{\alpha : u_*(\alpha) \leqslant x \leqslant u^*(\alpha)\}$$

is a fuzzy number with parameterization given by $\{(u_*(\alpha), u^*(\alpha), \alpha) : 0 \leqslant \alpha \leqslant 1\}$. Moreover, if $\tilde{u} : \mathbb{R} \to I$ is a fuzzy number with parameterization given by $\{(u_*(\alpha), u^*(\alpha), \alpha) : 0 \leqslant \alpha \leqslant 1\}$, then functions $u_*(\alpha)$ and $u^*(\alpha)$ satisfy conditions (1)–(5).

For $\tilde{u}, \tilde{v} \in \mathscr{F}$ the fuzzy addition and scalar multiplication can be defined, respectively, as: for $x \in \mathbb{R}$,

$$(\tilde{u} + \tilde{v})(x) = \sup_{y\in\mathbb{R}}[\min(\tilde{u}(y), \tilde{v}(x-y))],$$

and

$$(k\tilde{u})(x) = \begin{cases} \tilde{u}(x/k) & k > 0, \\ \tilde{0} & k = 0, \end{cases}$$

where $\tilde{0} \in \mathscr{F}$. We accept the subtraction of fuzzy numbers as defined by Dubois and Prade [4].

For this, define the opposite of a fuzzy number $\tilde{u}$ to be the fuzzy number $-\tilde{u}$ satisfying

$$(-\tilde{u})(x) = \tilde{u}(-x).$$

In other words, if $\tilde{u}$ is represented by the parametric form

$$\{(u_*(\alpha), u^*(\alpha), \alpha) : 0 \leqslant \alpha \leqslant 1\},$$

then $-\tilde{u}$ is represented by the corresponding parametric form

$$\{(-u^*(\alpha), -u_*(\alpha), \alpha) : 0 \leqslant \alpha \leqslant 1\}.$$

In this paper, if there is no confusion, we represent a fuzzy number $\tilde{u}$ as $\langle u_*(\alpha), u^*(\alpha)\rangle$ instead of the triple $(u_*(\alpha), u^*(\alpha), \alpha)$.

Definition 2.3 (*Triangular fuzzy number*). A fuzzy number $\tilde{a} = \langle a_*(\alpha), a^*(\alpha)\rangle$ is said to be a triangular fuzzy number if $a_*(1) = a^*(1)$. Moreover, if $a_*(\alpha)$ and $a^*(\alpha)$ are linear then we say $\tilde{a}$ a linear triangular fuzzy number. We denote a linear triangular fuzzy number by $\langle a_*(0), a_*(1), a^*(0)\rangle$.

For example for the fuzzy number $\tilde{a} = \langle 0, 1, 4\rangle$, we have $\tilde{a}[\alpha] = [\alpha, 4 - 3\alpha]$ for $\alpha \in [0, 1]$.

Definition 2.4. Let $\widetilde{A} = (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \in \mathscr{F}^n$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ be an n-dimensional fuzzy vector and an n-dimensional real vector, respectively. We define the product of a fuzzy vector with a real vector as $\widetilde{A}^t x = \sum_{i=1}^n \tilde{a}_i x_i$, which is a fuzzy number.

Definition 2.5. For $\tilde{u}, \tilde{v} \in \mathscr{F}$, we say that $\tilde{u} \preccurlyeq \tilde{v}$ if for each $\alpha \in [0, 1]$, $u_*(\alpha) \leqslant v_*(\alpha), u^*(\alpha) \leqslant v^*(\alpha)$. If $\tilde{u} \preccurlyeq \tilde{v}$, $\tilde{v} \preccurlyeq \tilde{u}$, then $\tilde{u} = \tilde{v}$. We say that $\tilde{u} \prec \tilde{v}$, if $\tilde{u} \preccurlyeq \tilde{v}$ and $\exists\ \alpha_0 \in [0, 1]$ such that $u_*(\alpha_0) < v_*(\alpha_0)$ or $u^*(\alpha_0) < v^*(\alpha_0)$. For $\tilde{u}, \tilde{v} \in \mathscr{F}$, if either $\tilde{u} \preccurlyeq \tilde{v}$ or $\tilde{v} \preccurlyeq \tilde{u}$, then we say that $\tilde{u}$ and $\tilde{v}$ are comparable, otherwise non-comparable.

Note that $\preccurlyeq$ is a partial order relation on $\mathscr{F}$. Sometimes we may write $\tilde{v} \succcurlyeq \tilde{u}$ instead of $\tilde{u} \preccurlyeq \tilde{v}$.

Lemma 2.6 (Yu–Ru Syau [13, Theorem 4.1]). *Let $\tilde{u}, \tilde{v} \in \mathscr{F}$ and $u[\alpha] = [u_*(\alpha), u^*(\alpha)]$, $v[\alpha] = [v_*(\alpha), v^*(\alpha)]$ for $\alpha \in [0, 1]$. Denote by*

$$\max\{u_*(\alpha), v_*(\alpha)\} = w_*(\alpha), \quad \max\{u^*(\alpha), v^*(\alpha)\} = w^*(\alpha)$$

for all $\alpha \in [0, 1]$. Then the family of $[w_*(\alpha), w^*(\alpha)]$ represents the α-level sets of a fuzzy number $\tilde{w} \in \mathscr{F}$. Moreover $\tilde{w}$ is the least upper bound(sup) of $\{\tilde{u}, \tilde{v}\}$. In a similar way infimum is also defined.

Definition 2.7 (*Buckley–Feuring* [3]). Let $\tilde{f}$, be a fuzzy mapping from the set of real numbers $\mathbb{R}$ to the set of all fuzzy numbers, let $\tilde{f}(t)[\alpha] = [f_*(t, \alpha), f^*(t, \alpha)]$. Assume that the partial derivatives of $f_*(t, \alpha), f^*(t, \alpha)$ with respect to $t \in \mathbb{R}$ for each $\alpha \in [0, 1]$ exist and are, respectively, denoted by $f_*'(t, \alpha), f^{*\prime}(t, \alpha)$. Let $\Gamma(t, \alpha) = [f_*'(t, \alpha), f^{*\prime}(t, \alpha)]$ for $t \in \mathbb{R}, \alpha \in [0, 1]$. If $\Gamma(t, \alpha)$ defines the α-cut of a fuzzy number for each $t \in \mathbb{R}$, then $\tilde{f}(t)$ is said to be differentiable and is written as $\frac{d\tilde{f}}{dt}[\alpha] = \Gamma(t, \alpha) = [f_*'(t, \alpha), f^{*\prime}(t, \alpha)]$, for all $t \in \mathbb{R}, \alpha \in [0, 1]$.

3. Minimization of a fuzzy mapping

Definition 3.1. Let $\tilde{f} : \Omega \subseteq \mathbb{R}^n \to \mathscr{F}$ be a fuzzy mapping. Consider the problem

$$\begin{aligned} &\text{Minimize} \quad \tilde{f}(x), \\ &\text{subject to} \quad x \in \Omega. \end{aligned}$$

A point $x \in \Omega$ is called a feasible solution. If $\hat{x} \in \Omega$ and for no $x \in \Omega$, $\tilde{f}(x) \prec \tilde{f}(\hat{x})$, then $\hat{x}$ is called an optimal solution, a global optimal solution, or simply a solution to the problem. If $\hat{x} \in \Omega$ and if there exists an ϵ-neighborhood $N_\epsilon(\hat{x})$ around $\hat{x}$ such that for no $x \in \Omega \cap N_\epsilon(\hat{x})$, $\tilde{f}(x) \prec \tilde{f}(\hat{x})$, then $\hat{x}$ is called a local optimal solution. Similarly, if $\hat{x} \in \Omega$ and if there exists an ϵ-neighborhood $N_\epsilon(\hat{x})$ around $\hat{x}$ for some $\epsilon > 0$ such that for no $x(\neq \hat{x}) \in \Omega \cap N_\epsilon(\hat{x})$, $\tilde{f}(x) \preccurlyeq \tilde{f}(\hat{x})$, then $\hat{x}$ is called a strict local optimal solution.

Throughout this paper we have accepted the fuzzy differentiability concept due to Buckley–Feuring [2,3].

Definition 3.2 (*Fuzzy mapping*). Let $\tilde{f} : \Omega \to \mathscr{F}$ be a fuzzy mapping, where $\Omega \subseteq \mathbb{R}^n$ and $\mathscr{F}$ is the set of fuzzy numbers. The α-cut of $\tilde{f}$ at $x \in \Omega$, which is a closed and bounded interval can be denoted by $\tilde{f}(x)[\alpha] = [f_*(x, \alpha), f^*(x, \alpha)]$ where $f_*(x, \alpha) = \min\{\tilde{f}(x)[\alpha]\}$ and $f^*(x, \alpha) = \max\{\tilde{f}(x)[\alpha]\}$. Thus, $\tilde{f}$ can be understood by the two functions $f_*(x, \alpha)$ and $f^*(x, \alpha)$, which are functions from $\Omega \times [0, 1]$ to the set of real numbers $\mathbb{R}$, $f_*(x, \alpha)$ is a bounded increasing function of α and $f^*(x, \alpha)$ is a bounded decreasing function of α. Moreover, $f_*(x, \alpha) \leqslant f^*(x, \alpha)$ for each $\alpha \in [0, 1]$.

Definition 3.3 (*Continuity of a fuzzy mapping*). Let $\tilde{f} : \Omega \subseteq \mathbb{R}^n \to \mathscr{F}$ be a fuzzy mapping. Then, $\tilde{f}$ is said to be continuous at $x \in \Omega$, if for each $\alpha \in [0, 1]$, both $f_*(x, \alpha), f^*(x, \alpha)$ are continuous functions of x.

Definition 3.4 (*Gradient of a fuzzy function*). Let $\tilde{f} : \Omega \to \mathscr{F}$ be a fuzzy mapping, where Ω is an open subset of $\mathbb{R}^n$. Let $x = (x_1, x_2, \ldots, x_n) \in \Omega$. Let D_{x_i}, $(i = 1, 2, \ldots, n)$ stand for the "partial differentiation" with respect to the ith variable x_i. Assume that for all $\alpha \in [0, 1]$, $f_*(x, \alpha), f^*(x, \alpha)$ have continuous partial derivatives so that $D_{x_i} f_*(x, \alpha), D_{x_i} f^*(x, \alpha)$ are continuous. Define

$$\tilde{D}_{x_i}\tilde{f}(x)[\alpha] = [D_{x_i} f_*(x, \alpha), D_{x_i} f^*(x, \alpha)] \quad \text{for } i = 1, 2, \ldots, n, \ \alpha \in [0, 1]. \tag{1}$$

If for each $i = 1, 2, \ldots, n$, (1) defines the α-cut of a fuzzy number, then we will say that $\tilde{f}$ is differentiable at x, and we write

$$\tilde{\nabla}\tilde{f}(x) = (\tilde{D}_{x_1}\tilde{f}(x), \tilde{D}_{x_2}\tilde{f}(x), \ldots, \tilde{D}_{x_n}\tilde{f}(x)). \tag{2}$$

We call $\tilde{\nabla}\tilde{f}(x)$, the gradient of the fuzzy function $\tilde{f}$ at x. Thus, from Lemma 2.2, the sufficient conditions that the gradient of $\tilde{f}$ at x exists are

for each $i = 1, 2, \ldots, n$, $\alpha \in [0, 1]$,

(i) the partial derivatives of $f_*(x,\alpha)$ and $f^*(x,\alpha)$ with respect to x_i exist,
(ii) $D_{x_i}f_*(x,\alpha)$ is a continuous increasing function of α,
(iii) $D_{x_i}f^*(x,\alpha)$ is a continuous decreasing function of α,
(iv) $D_{x_i}f_*(x,1) \leqslant D_{x_i}f^*(x,1)$.

Note that $\tilde{\nabla}\tilde{f}(x)$ is an n-dimensional fuzzy vector.

A fuzzy mapping $\tilde{f}$ is said to be differentiable at x if $\tilde{\nabla}\tilde{f}(x)$ exists and both $f_*(x,\alpha), f^*(x,\alpha)$ for each $\alpha \in [0,1]$ are differentiable at x.

Note: For the gradient of a fuzzy mapping we use the symbol $\tilde{\nabla}$, whereas for the gradient of a real valued function we use the symbol ∇.

Definition 3.5 (*Directional derivative of a fuzzy function*). Let $\tilde{f} : \Omega \subseteq \mathbb{R}^n \to \mathscr{F}$ be a fuzzy mapping. For $x \in \Omega$, let $d \in \mathbb{R}^n$ such that $x + \lambda d \in \Omega$ for $\lambda > 0$ and sufficiently small. The directional derivative of $\tilde{f}$ at x along the vector d (if it exists) is a fuzzy number denoted by $\tilde{f}'(x;d)$ and whose α-cut is defined as,

$$\tilde{f}'(x;d)[\alpha] = [f'_*((x;d),\alpha), f^{*\prime}((x;d),\alpha)],$$

where $f'_*((x;d),\alpha) = \lim_{\lambda\to 0+}\frac{f_*(x+\lambda d,\alpha)-f_*(x,\alpha)}{\lambda}$ and $f^{*\prime}((x;d),\alpha) = \lim_{\lambda\to 0+}\frac{f^*(x+\lambda d,\alpha)-f^*(x,\alpha)}{\lambda}$.

In the following example, we illustrate the above two concepts.

Example 3.6. Let $\tilde{f} : \mathbb{R}^2 \to \mathscr{F}$ be defined by

$$\tilde{f}(x_1,x_2) = \langle 0;2;4\rangle x_1^2 + \langle 0;2;4\rangle x_2^2 + \langle 1;3;5\rangle,$$
$$\tilde{f}(x_1,x_2)[\alpha] = [2\alpha, 4-2\alpha]x_1^2 + [2\alpha, 4-2\alpha]x_2^2 + [1+2\alpha, 5-2\alpha], \quad \alpha \in [0,1],$$

so we have, $f_*(x_1,x_2,\alpha) = 2\alpha x_1^2 + 2\alpha x_2^2 + (1+2\alpha)$ and $f^*(x_1,x_2,\alpha) = (4-2\alpha)x_1^2 + (4-2\alpha)x_2^2 + (5-2\alpha)$, $\alpha \in [0,1]$.

1. Now $D_{x_1}f_*(x_1,x_2,\alpha) = 4\alpha x_1, D_{x_1}f^*(x_1,x_2,\alpha) = 2(4-2\alpha)x_1, D_{x_2}f_*(x_1,x_2,\alpha) = 4\alpha x_2, D_{x_2}f^*(x_1,x_2,\alpha) = 2(4-2\alpha)x_2$. Thus, $\tilde{\nabla}\tilde{f}(x) = (\tilde{D}_{x_1}\tilde{f}(x), \tilde{D}_{x_2}\tilde{f}(x))$, where $\tilde{D}_{x_1}\tilde{f}(x) = \langle 4\alpha x_1, 2(4-2\alpha)x_1\rangle$, $\tilde{D}_{x_2}\tilde{f}(x) = \langle 4\alpha x_2, 2(4-2\alpha)x_2\rangle$. Notice that both $\tilde{D}_{x_1}\tilde{f}(x), \tilde{D}_{x_2}\tilde{f}(x)$ are fuzzy numbers for $x_1 \geqslant 0, x_2 \geqslant 0$. Thus, $\tilde{\nabla}\tilde{f}(x)$ exist in the first (non-negative) quadrant of $\mathbb{R}^2$.
2. Let $(x_1,x_2) = (1,2), (d_1,d_2) = (1,1)$. We find the directional derivative of $\tilde{f}$ at (x_1,x_2) along (d_1,d_2)

$$\begin{aligned}
f'_*(((x_1,x_2);(d_1,d_2)),\alpha) &= f'_*(((1,2);(1,1)),\alpha),\\
&= \lim_{\lambda\to 0+}\frac{2\alpha(1+\lambda)^2 + 2\alpha(2+\lambda)^2 + (1+2\alpha) - [2\alpha + 8\alpha + 1 + 2\alpha]}{\lambda},\\
&= \lim_{\lambda\to 0+}(4\alpha\lambda + 12\alpha),\\
&= 12\alpha,\\
f^{*\prime}(((x_1,x_2);(d_1,d_2)),\alpha) &= f^{*\prime}(((1,2);(1,1)),\alpha),\\
&= \lim_{\lambda\to 0+}\frac{(4-2\alpha)(1+\lambda)^2 + (4-2\alpha)(2+\lambda)^2 + (5-2\alpha) - [(4-2\alpha) + 4(4-2\alpha) + 5 - 2\alpha]}{\lambda},\\
&= 24 - 12\alpha.
\end{aligned}$$

Thus $\tilde{f}'((x_1,x_2);(d_1,d_2))[\alpha] = [12\alpha, 24-12\alpha]$.
Hence, the directional derivative of the above fuzzy function at (1, 2) in the direction (1, 1) exists and is a fuzzy number whose α-level set is $[12\alpha, 24-12\alpha]$.

In this paper, we have tried to use the concept of minimum of a convex fuzzy mapping whose range is the set of fuzzy numbers. Yu–Ru Syau [13] has defined maximum and minimum between two fuzzy numbers. Also Nanda–Kar [10] (in Convex Fuzzy Mappings), and Wu [15] (in Saddle Point Optimality conditions in Fuzzy Optimization Problems) have used a partial order relation in the set of fuzzy numbers to get the minimum value. But we see that in these approaches the minimum of a set of fuzzy numbers may not exist at a point in the domain of the fuzzy mapping. For non-linear fuzzy optimization problem we need a point where the minimum of the fuzzy mapping exists. This type of difficulty arises due to the fact that the set of fuzzy numbers is not totally ordered. This has forced us to define class of comparable and non-comparable fuzzy mapping.

Definition 3.7 (*Comparable fuzzy function*). Let $\tilde{f}: \Omega \subseteq \mathbb{R}^n \to \mathscr{F}$ be a fuzzy mapping. Then, $\tilde{f}$ is said to be a comparable fuzzy function if for each pair $x_1 \neq x_2 \in \Omega$, $\tilde{f}(x_1)$ and $\tilde{f}(x_2)$ are comparable. Otherwise, $\tilde{f}$ is said to be a non-comparable fuzzy function. Let $\mathscr{E}$ denote the set of all comparable fuzzy functions.

Example 3.8. The function $\tilde{f}$ defined in Example 3.6, that is,

$$\tilde{f}(x_1, x_2) = \langle 0;2;4\rangle x_1^2 + \langle 0;2;4\rangle x_2^2 + \langle 1;3;5\rangle$$

is an example of a comparable fuzzy function.

Consider another example.

Example 3.9. Let $\tilde{f}: \Omega \subseteq \mathbb{R} \to \mathscr{F}$, where Ω is the set of all positive real numbers, be defined by

$$\tilde{f}(x) = \langle 0;1;2\rangle x^2 + \langle 1;2;3\rangle x + \langle 1;2;4\rangle,$$
$$\tilde{f}(x)[\alpha] = [\alpha, 2-\alpha]x^2 + [1+\alpha, 3-\alpha]x + [1+\alpha, 4-2\alpha], \quad \alpha \in [0,1].$$

If $x \in \mathbb{R}^+$, then

$$f_*(x,\alpha) = \alpha x^2 + (1+\alpha)x + (1+\alpha)$$

and

$$f^*(x,\alpha) = (2-\alpha)x^2 + (3-\alpha)x + (4-2\alpha), \quad \alpha \in [0,1].$$

For $x > y \in \mathbb{R}^+$, we have

$$f_*(x,\alpha) - f_*(y,\alpha) = \alpha(x^2 - y^2) + (1+\alpha)(x-y) > 0, \tag{3}$$
$$f^*(x,\alpha) - f^*(y,\alpha) = (2-\alpha)(x^2 - y^2) + (3-\alpha)(x-y) > 0. \tag{4}$$

Thus, $\tilde{f}(y) \preceq \tilde{f}(x)$. Hence, $\tilde{f}$ is a comparable fuzzy function.

Below we have furnished an example of a fuzzy mappings for which Yu–Ru Syau [13] and other's approach to find a point of minimum fails. This fuzzy mapping is a non-comparable fuzzy mapping, hence all the points are point of minimum.

Example 3.10. Consider the fuzzy mapping, $\tilde{h}: \Omega \subseteq \mathbb{R} \to \mathscr{F}$, where Ω is the set of all positive real numbers,

$$\tilde{h}(x) = \langle 0;1;4\rangle x^2 - \langle 0;3;4\rangle x + \langle 1;2;4\rangle.$$

The α-cut is given by

$$\tilde{h}(x)[\alpha] = [\alpha, 4-3\alpha]x^2 - [3\alpha, 4-\alpha]x + [1+\alpha, 4-2\alpha].$$

Hence,

$$h_*(x,\alpha) = \alpha x^2 - (4-\alpha)x + 1 + \alpha$$

and

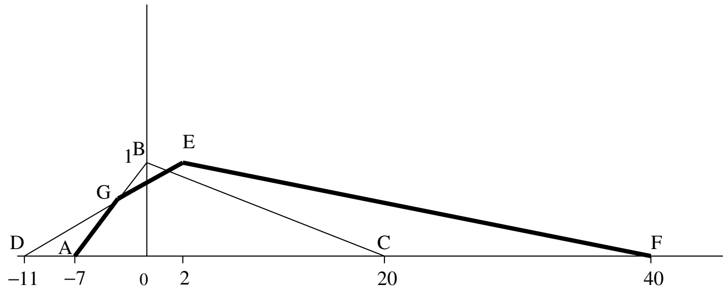
$$h^*(x,\alpha) = (4-3\alpha)x^2 - 3\alpha x + (4-2\alpha).$$

Now let $x_1, x_2 \in \mathbb{R}$ such that $x_1 > x_2 > 0$. Then

$$h_*(x_1,\alpha) - h_*(x_2,\alpha) = \alpha(x_1^2 - x_2^2) - (4-\alpha)(x_1 - x_2) = (x_1 - x_2)[\alpha(x_1 + x_2) - (4-\alpha)], \tag{5}$$
$$h^*(x_1,\alpha) - h^*(x_2,\alpha) = (4-3\alpha)(x_1^2 - x_2^2) - 3\alpha(x_1 - x_2) = (x_1 - x_2)[(4-3\alpha)(x_1 + x_2) - 3\alpha]. \tag{6}$$

To check whether $\tilde{h}$ is comparable or not we have to check whether both (5) and (6) are simultaneously non-positive or simultaneously non-negative. Now as $\alpha \to 0$, $\frac{4-\alpha}{\alpha} \to \infty$ and $\frac{3\alpha}{4-3\alpha} \to 0$. Therefore, if both (5) and (6) non-negative then $(x_1 + x_2)$ is infinite, and if (5) and (6) non-positive then $(x_1 + x_2)$ becomes non-positive, this contradicts that x_1, x_2 are finite positive real numbers. Hence, $\tilde{h}(x_1)$ and $\tilde{h}(x_2)$ are not comparable for all positive numbers $x_1 \neq x_2$.

Let $x_1 = 2, x_2 = 3$. Then $h_*(2,\alpha) = 7\alpha - 7$ and $h^*(2,\alpha) = 20 - 20\alpha$. $h_*(3,\alpha) = 13\alpha - 11$ and $h^*(3,\alpha) = 40 - 38\alpha$. Thus, $\tilde{h}(2) = \langle -7; 0; 20\rangle$, $\tilde{h}(3) = \langle -11; 2; 40\rangle$. Clearly, $\tilde{h}(2)$ and $\tilde{h}(3)$ are not comparable. Hence, $\tilde{h}$ is a non-comparable fuzzy function.



Let $\tilde{h}(2) = \tilde{u}$(say) (the part ABC in figure) and $\tilde{h}(3) = \tilde{v}$(say) (the part DEF in figure).

Let $\tilde{h}(2) = \tilde{u}$(say) and $\tilde{h}(3) = \tilde{v}$(say). Let us find the supremum and infimum between $\tilde{u}$ and $\tilde{v}$ according to (Syau[13]) as follows:

$$\tilde{w} = \sup\{\tilde{u}, \tilde{v}\} = \bigcup_{\alpha\in[0,2/3]} [7\alpha - 7, 40 - 38\alpha] \bigcup_{\alpha\in[2/3,1]} [13\alpha - 11, 40 - 38\alpha]$$

and

$$\tilde{z} = \inf\{\tilde{u}, \tilde{v}\} = \bigcup_{\alpha\in[0,2/3]} [13\alpha - 11, 20 - 20\alpha] \bigcup_{\alpha\in[2/3,1]} [7\alpha - 7, 20 - 20\alpha].$$

In the figure, AGEF represents the fuzzy number $\tilde{w}$ and DGBC represents the fuzzy number $\tilde{z}$.

But there does not exist any $x \in \Omega$ such that $\tilde{h}(x) = \tilde{w}$ or for that matter $\tilde{h}(x) = \tilde{z}$.

Therefore, we will not consider the infimum (or supremum) in the minimization (or maximization) of a fuzzy mapping according to Yu–Ru Syau. Rather we will consider the order relation defined by Nanda–Kar [10], and Wu [15].

Remark 3.11. Since $\preccurlyeq$ is a partial order relation, minimum value of a non-comparable fuzzy function may not be unique. Hence, a non-comparable fuzzy function has a set of minimal values. Therefore, application of fuzzy differential calculus to find minimum of a fuzzy mapping may not be fruitful for a non-comparable fuzzy function. Explicitly, we cannot put the necessary condition for the minimum of a non-comparable differentiable fuzzy function as $\tilde{\nabla}\tilde{f}(x) = \tilde{0}$ which we can do for comparable fuzzy mappings (see Theorem 3.13). To overcome this type of difficulties we consider the fuzzy functions which are comparable. Otherwise for arbitrary fuzzy functions which are differentiable we may assume a point to be a point of minimum only if that is, $\tilde{\nabla}\tilde{f}(x) = \tilde{0}^*$ holds, where $\tilde{0}^*$ is a fuzzy number whose core is $\{0\}$. For example, for the fuzzy function $\tilde{h}$ in Example 3.10, $\tilde{\nabla}\tilde{h}(3/2) = \langle -4, 0, 12\rangle$.

Theorem 3.12. *Let $\tilde{f} : \Omega \subseteq \mathbb{R}^n \to \mathscr{F}$, ($\tilde{f} \in \mathscr{E}$ and Ω an open set) be differentiable at $x \in \Omega$. If there is a vector $d \in \mathbb{R}^n$ satisfying $\nabla f_*(x,\alpha)^t d < 0$, or $\nabla f^*(x,\alpha)^t d < 0$, for at least one $\alpha \in [0,1]$ implies there exists a $\delta > 0$ satisfying $\tilde{f}(x + \lambda d) \preccurlyeq \tilde{f}(x)$ for each $\lambda \in (0, \delta)$, in that case we say that d is a descent direction of $\tilde{f}$ at x.*

Proof. By the differentiability of $\tilde{f}$ at x, we have $f_*(.,\alpha)$ and $f^*(.,\alpha)$ are also differentiable at x. Without loss of generality assume that $\nabla f^*(x,\alpha_0)^t d < 0$, for some $\alpha_0 \in [0,1]$. Then, by Theorem 4.1.2 (Bazaraa [1]) there exists a $\delta > 0$ such that $f^*(x+\lambda d,\alpha_0) < f^*(x,\alpha_0)$ for $\lambda \in (0,\delta)$. Now since $\tilde{f} \in \mathscr{E}$, we have $\tilde{f}(x+\lambda d) \preccurlyeq \tilde{f}(x)$ or $\tilde{f}(x) \preccurlyeq \tilde{f}(x+\lambda d)$. But since $f^*(x+\lambda d,\alpha_0) < f^*(x,\alpha_0)$, $\tilde{f}(x) \preccurlyeq \tilde{f}(x+\lambda d)$ is not true. Thus, $\tilde{f}(x+\lambda d) \prec \tilde{f}(x)$. □

Here, we present the first-order condition for a local minimum of a fuzzy mapping.

Theorem 3.13. *Let $\tilde{f} \in \mathscr{E}$ be differentiable at $x \in \Omega \subseteq \mathbb{R}^n$ an open set. If x is a point of local minimum, then $\tilde{\nabla}\tilde{f}(x) = \tilde{0}$.*

Proof. Suppose $\tilde{\nabla}\tilde{f}(x) \neq \tilde{0}$. Then there exists $\alpha_0 \in [0,1]$, such that $\nabla f_*(x,\alpha_0) \neq 0$ or $\nabla f^*(x,\alpha_0) \neq 0$. With out loss of generality suppose that $\nabla f_*(x,\alpha_0) \neq 0$. Let $d = -\nabla f_*(x,\alpha_0)$. Then we get $\nabla f_*(x,\alpha_0)^t d = -\|\nabla f_*(x,\alpha_0)\|^2 < 0$. Now by Theorem 3.12, there exists a $\delta > 0$ such that $\tilde{f}(x+\lambda d) \prec \tilde{f}(x)$ for $\lambda \in (0,\delta)$, which contradicts the assumption that x is a point of local minimum. Therefore, $\tilde{\nabla}\tilde{f}(x) = \tilde{0}$. □

Example 3.14. Consider the fuzzy mapping $\tilde{f}$ in Example 3.6. $\tilde{\nabla}\tilde{f}(x) = \tilde{0} \Rightarrow \tilde{D}_{x_1}\tilde{f}(x) = \tilde{0}, \tilde{D}_{x_2}\tilde{f}(x) = \tilde{0} \Rightarrow x_1 = 0, x_2 = 0$.

Thus, at $(0,0) \in \mathbb{R}^2$, $\tilde{\nabla}\tilde{f}(0,0) = \tilde{0}$ and it can be easily seen that $\tilde{f}$ has minimum at $(0,0)$.

Definition 3.15 (*Twice differentiable fuzzy function and Hessian of a fuzzy function*). Let $\tilde{f} : \Omega \to \mathscr{F}$ be a fuzzy mapping, where Ω is an open subset of $\mathbb{R}^n$. Let $x = (x_1, x_2, \ldots, x_n) \in \Omega$. Let $D_{x_i x_j}$, $(i,j = 1,2,\ldots,n)$ stand for the "second-order partial" with respect to the i-th variable x_i and jth variable x_j. Assume that $\tilde{\nabla}\tilde{f}(x)$ exists and for all $\alpha \in [0,1]$, $f_*(x,\alpha), f^*(x,\alpha)$ have continuous second-order partial derivatives so that $D_{x_i x_j} f_*(x,\alpha)$, $D_{x_i x_j} f^*(x,\alpha)$ are continuous (here $D_{x_i x_j} f_*(x,\alpha) = D_{x_i}(D_{x_j} f_*(x,\alpha)) = D_{x_j}(D_{x_i} f_*(x,\alpha)) = D_{x_j x_i} f_*(x,\alpha)$, $D_{x_i x_j} f^*(x,\alpha) = D_{x_i}(D_{x_j} f^*(x,\alpha)) = D_{x_j}(D_{x_i} f^*(x,\alpha)) = D_{x_j x_i} f^*(x,\alpha)$). Define

$$\tilde{D}_{x_i x_j}\tilde{f}(x)[\alpha] = [D_{x_i x_j} f_*(x,\alpha), D_{x_i x_j} f^*(x,\alpha)] \quad \text{for } i,j = 1,2,\ldots,n, \;\; \alpha \in [0,1]. \tag{7}$$

If for each $i,j = 1,2,\ldots,n$, (7) defines the α-cut of a fuzzy number, then we define the Hessian of the fuzzy function (in the matrix notation) as follows:

$$\tilde{\nabla}^2\tilde{f}(x) = (\tilde{D}_{x_i x_j}\tilde{f}(x))_{i,j=1,2,\ldots,n}. \tag{8}$$

We will say that $\tilde{f}$ is twice differentiable at x, if the Hessian of the fuzzy function exists and both $f_*(x,\alpha), f^*(x,\alpha)$ are twice differentiable at x. The sufficient conditions that the Hessian of the fuzzy mapping $\tilde{f}$ at x exists are

for each $i,j = 1,2,\ldots,n$, $\alpha \in [0,1]$,

(i) $\tilde{\nabla}\tilde{f}(x)$ exists,
(ii) the second-order partial derivatives of $f_*(x,\alpha)$ and $f^*(x,\alpha)$ with respect to x_i, x_j exist,
(iii) $D_{x_i x_j} f_*(x,\alpha)$ is a continuous increasing function of α,
(iv) $D_{x_i x_j} f^*(x,\alpha)$ is a continuous decreasing function of α,
(v) $D_{x_i x_j} f_*(x,1) \leqslant D_{x_i x_j} f^*(x,1)$.

Definition 3.16 (*Positive definite and positive semi-definite fuzzy matrix*). Let $\widetilde{A} = (\tilde{a}_{ij})$ be an $n \times n$ fuzzy matrix in which the entries $\tilde{a}_{ij}$ $(i,j = 1,2,\ldots,n)$ are all fuzzy numbers. $\widetilde{A}$ is said to be a positive definite fuzzy matrix if for each $\alpha \in [0,1]$, the left hand and the right-hand α-level matrices $((a_*)_{ij}(\alpha))$ and $((a^*)_{ij}(\alpha))$ (where $\tilde{a}_{ij} = \langle (a_*)_{ij}(\alpha), (a^*)_{ij}(\alpha) \rangle$), are all positive definite. Similarly, $\widetilde{A}$ is said to be a positive semi-definite fuzzy matrix if for each $\alpha \in [0,1]$, the matrices $((a_*)_{ij}(\alpha))$ and $((a^*)_{ij}(\alpha))$ (where $\tilde{a}_{ij} = \langle (a_*)_{ij}(\alpha), (a^*)_{ij}(\alpha) \rangle$), are all positive semi-definite.

Example 3.17. Consider the fuzzy matrix $\widetilde{A} = \begin{pmatrix} \tilde{a} & \tilde{0} \\ \tilde{0} & \tilde{a} \end{pmatrix}$ where $\tilde{a} = \langle 0, 4, 8 \rangle = \langle 4\alpha, 8 - 4\alpha \rangle$. We will show that this matrix is a positive semi-definite fuzzy matrix but not positive definite fuzzy matrix. For this we consider the α-level matrices $\begin{pmatrix} 4\alpha & 0 \\ 0 & 4\alpha \end{pmatrix}$ and $\begin{pmatrix} 8-4\alpha & 0 \\ 0 & 8-4\alpha \end{pmatrix}$. It can be easily checked that both the above matrices are positive semi-definite for all $\alpha \in [0, 1]$. But for $\alpha = 0$, the first matrix is not positive definite. Hence, $\widetilde{A}$ is a positive semi-definite fuzzy matrix but not positive definite fuzzy matrix.

Theorem 3.18. *Let $\tilde{f} \in \mathscr{E}$ be twice differentiable at $x \in \Omega$ an open set. If x is a point of local minimum, then $\tilde{\nabla}\tilde{f}(x) = \tilde{0}$ and $\tilde{\nabla}^2\tilde{f}(x)$ is fuzzy positive semi-definite.*

Proof. Consider an arbitrary direction $d \in \mathbb{R}^n$. Then, since $\tilde{f}$ is twice differentiable at x, we have $f_*(., \alpha)$ and $f^*(., \alpha)$ are twice differentiable at x for all $\alpha \in [0, 1]$, and so by Taylor's expansion of $f_*(., \alpha)$ and $f^*(\cdot, \alpha)$ at x, we have

$$f_*(x + \lambda d, \alpha) = f_*(x, \alpha) + \lambda \nabla f_*(x, \alpha)^t d + (1/2)\lambda^2 d^t \nabla^2 f_*(x, \alpha) d + \lambda^2 \|d\|^2 \theta(x; \lambda d, \alpha), \tag{9}$$

$$f^*(x + \lambda d, \alpha) = f^*(x, \alpha) + \lambda \nabla f^*(x, \alpha)^t d + (1/2)\lambda^2 d^t \nabla^2 f^*(x, \alpha) d + \lambda^2 \|d\|^2 \theta(x; \lambda d, \alpha), \tag{10}$$

where $\theta(x; \lambda d, \alpha) \to 0$ as $\lambda \to 0$. Since x is a point of local minimum, by Theorem 3.13 we have $(\tilde{\nabla}\tilde{f}(x) = \tilde{0}$, that is, we have $\nabla f_*(x, \alpha) = 0 = \nabla f^*(x, \alpha)$ for each $\alpha \in [0, 1]$. Now by rearrangement of (9) and (10) and dividing by $\lambda^2 > 0$ individually, we get

$$\frac{f_*(x + \lambda d, \alpha) - f_*(x, \alpha)}{\lambda^2} = (1/2) d^t \nabla^2 f_*(x, \alpha) d + \|d\|^2 \theta(x; \lambda d, \alpha), \tag{11}$$

$$\frac{f^*(x + \lambda d, \alpha) - f^*(x, \alpha)}{\lambda^2} = (1/2) d^t \nabla^2 f^*(x, \alpha) d + \|d\|^2 \theta(x; \lambda d, \alpha). \tag{12}$$

Since x is a point of local minimum, we have $f_*(x + \lambda d, \alpha) \geqslant f_*(x, \alpha)$ and $f^*(x + \lambda d, \alpha) \geqslant f^*(x, \alpha)$ for $\alpha \in [0, 1]$ for λ sufficiently small. Therefore, the right side of (11) and (12) are positive and by taking the limits as $\lambda \to 0$ the result follows. $\square$

Example 3.19. For the fuzzy mapping $\tilde{f}$ defined in Example 3.6, it can be easily calculated that $\tilde{\nabla}^2\tilde{f}(x)$ exists and is equal to $\begin{pmatrix} \tilde{a} & \tilde{0} \\ \tilde{0} & \tilde{a} \end{pmatrix}$ which is a positive semi-definite fuzzy matrix as seen in Example 3.17.

4. Convexity and generalized convexity of fuzzy mappings

Definition 4.1 (*Convex fuzzy mapping*). Let $\tilde{f} : \Omega \subseteq \mathbb{R}^n \to \mathscr{F}$ be a fuzzy mapping, where Ω is a convex subset of $\mathbb{R}^n$. $\tilde{f}$ is said to be convex on Ω, if for each $\alpha \in [0, 1]$ both $f_*(x, \alpha), f^*(x, \alpha)$ are convex on Ω, that is, for $0 \leqslant \lambda \leqslant 1$, $x, y \in \Omega$,

1. $f_*((1 - \lambda)x + \lambda y, \alpha) \leqslant (1 - \lambda) f_*(x, \alpha) + \lambda f_*(y, \alpha)$ and
2. $f^*((1 - \lambda)x + \lambda y, \alpha) \leqslant (1 - \lambda) f^*(x, \alpha) + \lambda f^*(y, \alpha)$.

$\tilde{f}$ is said to be concave if $-\tilde{f}$ is convex.

Example 4.2. The fuzzy function $\tilde{f}$ in Example 3.6 is convex on $\mathbb{R}^2$, and the fuzzy function $\tilde{h}$ in Example 3.10 is convex on Ω.

In the next two results, we discuss the necessary and sufficient conditions for a differentiable fuzzy mapping to be convex.

Theorem 4.3. *Let $\tilde{f}$ be a fuzzy mapping on an open convex set $\Omega \subseteq \mathbb{R}^n$. Let $\tilde{f}$ be differentiable at $x_0 \in \Omega$. If $\tilde{f}$ is convex on Ω, then for each $x \in \Omega$ and $\alpha \in [0, 1]$, we have*

$$f_*(x,\alpha) - f_*(x_0,\alpha) \geqslant \nabla f_*(x_0,\alpha)^t(x - x_0), \tag{13}$$

$$f^*(x,\alpha) - f^*(x_0,\alpha) \geqslant \nabla f^*(x_0,\alpha)^t(x - x_0). \tag{14}$$

Proof. Let $\tilde{f}$ be convex at x_0. Then, $f_*(x,\alpha)$ and $f^*(x,\alpha)$ are convex at x_0 for each $\alpha \in [0,1]$. Therefore, by Theorem 6.1.1 (Mangasarian [9]), we have the desired result. □

Theorem 4.4. *Let $\tilde{f}$ be a fuzzy mapping from an open set $\Omega \subseteq \mathbb{R}^n$ to $\mathscr{F}$. Let $\tilde{f}$ be differentiable on Ω. Then, a necessary and sufficient condition for $\tilde{f}$ to be convex on Ω is that for each $x, y \in \Omega$ and $\alpha \in [0,1]$, the following two inequalities hold*:

$$[\nabla f_*(x,\alpha) - \nabla f_*(y,\alpha)]^t(x - y) \geqslant 0, \tag{15}$$

$$[\nabla f^*(x,\alpha) - \nabla f^*(y,\alpha)]^t(x - y) \geqslant 0. \tag{16}$$

Proof. A real valued differentiable function g defined on an open convex set Ω is convex if and only if for each $x, y \in \Omega$

$$[\nabla g(x) - \nabla g(y)]^t(x - y) \geqslant 0. \tag{17}$$

Since $\tilde{f}$ is a differentiable convex function on Ω, so $f_*(x,\alpha)$ and $f^*(x,\alpha)$ are real valued differentiable convex function on Ω for each $\alpha \in [0,1]$. Therefore, they satisfy (15) and (16), respectively.

Conversely, suppose $f_*(x,\alpha)$ and $f^*(x,\alpha)$ satisfy (15) and (16), respectively, for each $x, y \in \Omega$ and $\alpha \in [0,1]$. By assumption $\tilde{f}$ is differentiable on Ω and hence $f_*(x,\alpha), f^*(x,\alpha)$ are differentiable function on Ω for each $\alpha \in [0,1]$. Now following (17), both $f_*(x,\alpha), f^*(x,\alpha)$ are convex for each $\alpha \in [0,1]$. Hence, by definition $\tilde{f}$ is convex. □

The concept of directional derivative of a fuzzy mapping in a direction is important to find the optimal solution of fuzzy nonlinear programming problem. The following theorem proves the existence of directional derivative of a fuzzy mapping at a particular point.

Theorem 4.5. *Let Ω be an open convex subset of $\mathbb{R}^n$ and let $\tilde{f} : \Omega \to \mathscr{F}$ be a convex fuzzy mapping. For $x \in \Omega$ let the α-cut of $\tilde{f}$ at x be denoted by $\tilde{f}(x)[\alpha] = [f_*(x,\alpha), f^*(x,\alpha)]$. Then, for $d \neq 0 \in \mathbb{R}^n$, $f_*((x;d),\alpha)$ and $f^*((x;d),\alpha)$ exist for each $\alpha \in [0,1]$. Moreover, if $\langle f_*((x;d),\alpha), f^*((x;d),\alpha)\rangle$ represents a fuzzy number then the directional derivative of $\tilde{f}$ at x in the direction d that is, $\tilde{f}'(x;d)$ exists.*

Proof. Since $\tilde{f}$ is convex on Ω, we have $f_*(x,\alpha), f^*(x,\alpha)$ are convex functions on Ω for each $\alpha \in [0,1]$. Therefore, $f'_*((x;d),\alpha)$ and $f^{*\prime}((x;d),\alpha)$ exist for each $\alpha \in [0,1]$. Hence, the proof is complete. □

Theorem 4.6. *Let $\tilde{f}$ be a twice differentiable fuzzy mapping on an open convex set $\Omega \subseteq \mathbb{R}^n$ to $\mathscr{F}$. $\tilde{f}$ is convex on Ω if and only if for each $x \in \Omega$, $\tilde{\nabla}^2\tilde{f}(x)$ is a positive semi-definite fuzzy matrix.*

Proof. *Necessity*. Since $\tilde{f}$ is twice differentiable on Ω, we have for each $\alpha \in [0,1]$, $f_*(.,\alpha)$, and $f^*(.,\alpha)$ are twice differentiable on Ω. Assume that $\tilde{f}$ be convex on Ω, that is for each $\alpha \in [0,1]$, $f_*(.,\alpha)$, and $f^*(.,\alpha)$ are convex on Ω. Therefore, the Hessian matrices $\nabla^2 f_*(x,\alpha), \nabla^2 f^*(x,\alpha)$ for each $x \in \Omega, \alpha \in [0,1]$ are all positive semi-definite matrices. Since $\tilde{f}$ is twice differentiable, $\nabla^2 f_*(x,\alpha), \nabla^2 f^*(x,\alpha)$ are nothing but the left hand and right-hand end points of the α-cut of $\tilde{\nabla}^2\tilde{f}(x)$ for each $\alpha \in [0,1]$. Thus, $\tilde{\nabla}^2\tilde{f}(x)$ is a positive semi-definite fuzzy matrix.

Sufficiency. Since $\tilde{f}$ is twice differentiable, $\tilde{\nabla}^2\tilde{f}(x)$ exists for each $x \in \Omega$, moreover let $\tilde{\nabla}^2\tilde{f}(x)$ be a positive semi-definite fuzzy matrix. Then the left and right-hand α-level matrices of $\tilde{\nabla}^2\tilde{f}(x)$ (that is, $\nabla^2 f_*(x,\alpha), \nabla^2 f^*(x,\alpha)$ for each $\alpha \in [0,1]$, respectively), are all positive semi-definite matrices. As a result $f_*(x,\alpha)$ and $f^*(x,\alpha)$ are all convex on Ω for each $\alpha \in [0,1]$. Hence by definition $\tilde{f}$ is convex on Ω. □

Example 4.7. The fuzzy mapping $\tilde{f}$ in Example 3.6, is a convex fuzzy mapping since $\tilde{\nabla}^2\tilde{f}(x)$ is a positive semi-definite fuzzy matrix.

The concept of quasiconvex fuzzy mapping have been introduced by Nanda and Kar [10]. But the concept for finding the maximum of two fuzzy numbers has not been discussed in their paper. It may happen that two fuzzy numbers are not comparable. So we modify the definition of quasiconvex fuzzy mapping as follows:

Definition 4.8 (*Quasiconvex fuzzy mapping*). Let $\tilde{f} : \Omega \subseteq \mathbb{R}^n \to \mathscr{F}$ be a fuzzy mapping. Let Ω be a non-empty convex set in $\mathbb{R}^n$. $\tilde{f}$ is said to be quasiconvex if, for each $x, y \in \Omega$, the following inequality is true:

$$\tilde{f}[\lambda x + (1-\lambda)y] \preccurlyeq \max\{\tilde{f}(x), \tilde{f}(y)\} \quad \text{for each } \lambda \in (0,1), \tag{18}$$

whenever $\tilde{f}(x)$ and $\tilde{f}(y)$ are comparable.

The fuzzy mapping $\tilde{f}$ is said to be quasiconcave if $-\tilde{f}$ is quasiconvex.

Definition 4.9 (*Strictly quasiconvex fuzzy mapping*). Let $\tilde{f} : \Omega \subseteq \mathbb{R}^n \to \mathscr{F}$ be a fuzzy mapping. Let Ω be a non-empty convex set in $\mathbb{R}^n$. $\tilde{f}$ is said to be strictly quasiconvex if, for each $x, y \in \Omega$ with $\tilde{f}(x) \neq \tilde{f}(y)$, the following inequality is true:

$$\tilde{f}[\lambda x + (1-\lambda)y] \prec \max\{\tilde{f}(x), \tilde{f}(y)\} \quad \text{for each } \lambda \in (0,1), \tag{19}$$

whenever $\tilde{f}(x)$ and $\tilde{f}(y)$ are comparable.

The fuzzy mapping $\tilde{f}$ is said to be strictly quasiconcave if $-\tilde{f}$ is strictly quasiconvex.

Definition 4.10 (*Strongly quasiconvex fuzzy mapping*). Let $\tilde{f} : \Omega \subseteq \mathbb{R}^n \to \mathscr{F}$ be a fuzzy mapping. Let Ω be a non-empty convex set in $\mathbb{R}^n$. $\tilde{f}$ is said to be strongly quasiconvex if, for each $x, y \in \Omega$ with $x \neq y$, the following inequality is true:

$$\tilde{f}[\lambda x + (1-\lambda)y] \prec \max\{\tilde{f}(x), \tilde{f}(y)\} \quad \text{for each } \lambda \in (0,1), \tag{20}$$

whenever $\tilde{f}(x)$ and $\tilde{f}(y)$ are comparable.

The fuzzy mapping $\tilde{f}$ is said to be strongly quasiconcave if $-\tilde{f}$ is strongly quasiconvex.

Here, it may be noted that every strictly quasiconvex fuzzy mapping may not be quasiconvex. We show it by an example.

Example 4.11. Consider the fuzzy mapping $\tilde{f} : \mathbb{R} \to \mathscr{F}$ defined by

$$\tilde{f}(t) = \begin{cases} \tilde{b} & t = 0, \\ \tilde{0} & t \neq 0, \end{cases}$$

where $\tilde{b} = \langle 1, 2, 3 \rangle$. Then it can be easily checked that $\tilde{f}$ is strictly quasiconvex.

However, $\tilde{f}$ is not quasiconvex, since for $t_1 = 1, t_2 = -1$, we have $\tilde{f}(t_1) = \tilde{f}(t_2) = \tilde{0}$, but $\tilde{f}\left(\frac{1}{2}t_1 + \frac{1}{2}t_2\right) = \tilde{f}(0) = \tilde{b} \succ \max\{\tilde{f}(t_1), \tilde{f}(t_2)\}$.

Nanda and Kar [10] have proved that for a strictly quasiconvex fuzzy mapping $\tilde{f}$ defined on a convex subset Ω in $\mathbb{R}^n$, a local minimum is a global minimum. In the proof they used the concept of a local minimum as: Let $\hat{x}$ be a point of local minimum of $\tilde{f}$ in Ω means there exists a neighborhood N of $\hat{x}$ in Ω such that $\tilde{f}(\hat{x}) \preccurlyeq \tilde{f}(x)$ for all $x \in N$. But in case of fuzzy mapping it might happen that there exists no neighborhood N of $\hat{x}$ such that $\tilde{f}(\hat{x}) \preccurlyeq \tilde{f}(x)$ for all $x \in N$ (see Example 3.10). But still $\hat{x}$ is a point of local minimum if for no $x \in \Omega \cap N_\epsilon(\hat{x})$, $\tilde{f}(x) \prec \tilde{f}(\hat{x})$ for all $x \in N$. Hence, we have modified the result in [10] as follows:

Theorem 4.12. *Let $\tilde{f} : \Omega \subseteq \mathbb{R}^n \to \mathscr{F}$ be a strictly quasiconvex fuzzy mapping, where Ω is a nonempty open convex set in $\mathbb{R}^n$. If $\hat{x}$ is a local optimal solution, then there does not exist $x_0 \in \Omega$ such that $\tilde{f}(x_0) \prec \tilde{f}(\hat{x})$.*

Proof. Suppose there exists $x_0 \in \Omega$ such that $\tilde{f}(x_0) \prec \tilde{f}(\hat{x})$. By the convexity of Ω, $\lambda x_0 + (1-\lambda)\hat{x} \in \Omega$ for each $\lambda \in (0,1)$. Since $\hat{x}$ is a point of local minimum by assumption, then $\exists$ no $\tilde{f}[\lambda x_0 + (1-\lambda)\hat{x}] \prec \tilde{f}(\hat{x})$ for each $\lambda \in (0,\delta)$ and for some $\delta \in (0,1)$. But because $\tilde{f}$ is strictly quasiconvex and $\tilde{f}(x_0) \prec \tilde{f}(\hat{x})$, $\tilde{f}[\lambda x_0 + (1-\lambda)\hat{x}] \prec \tilde{f}(\hat{x})$ for each $\lambda \in (0,1)$. This contradicts that $\hat{x}$ is a local optimal point, and the proof is complete. $\square$

Theorem 4.13. *Let $\tilde{f} : \Omega \subseteq \mathbb{R}^n \to \mathscr{F}$ be a strongly quasiconvex fuzzy mapping, where Ω is a nonempty open convex set in $\mathbb{R}^n$. If $\hat{x}$ is a local optimal solution, then there exists no $x_0(\neq \hat{x}) \in \Omega$ such that $\tilde{f}(x_0) \preccurlyeq \tilde{f}(\hat{x})$. That is $\hat{x}$ is a unique global optimal solution with objective value $\tilde{f}(\hat{x})$.*

Proof. Since $\hat{x}$ is a local optimal solution, there exists an ϵ-neighborhood $N_\epsilon(\hat{x})$ around $\hat{x}$ such that for no $x \in \Omega \cap N_\epsilon(\hat{x})$, $\tilde{f}(x) \prec \tilde{f}(\hat{x})$. Contrary to the theorem, let $\exists x_0 \in \Omega$ with $x_0 \neq \hat{x}$ and $\tilde{f}(x_0) \preccurlyeq \tilde{f}(\hat{x})$. By strong quasiconvexity, it follows that

$$\tilde{f}[\lambda x_0 + (1-\lambda)\hat{x}] \prec \max\{\tilde{f}(x_0), \tilde{f}(\hat{x})\} = \tilde{f}(\hat{x}) \quad \text{for each } \lambda \in (0,1).$$

But for λ small enough, $\lambda x_0 + (1-\lambda)\hat{x} \in \Omega \cap N_\epsilon(\hat{x})$, and thus the above inequality contradicts that $\hat{x}$ is a local optimal point. Hence, the proof is complete. □

Below we derive the necessary and sufficient condition for a differentiable fuzzy mapping to be quasiconvex.

Theorem 4.14. *$\tilde{f} : \Omega \subseteq \mathbb{R}^n \to \mathscr{F}$ be a differentiable fuzzy mapping, where Ω is a nonempty open convex set in $\mathbb{R}^n$. Let $\tilde{f} \in \mathscr{E}$. Then $\tilde{f}$ is quasiconvex if and only if the following statement holds: If $x_1, x_2 \in \Omega$ and $\tilde{f}(x_1) \preccurlyeq \tilde{f}(x_2)$, then $\tilde{\nabla} \tilde{f}(x_2)^t(x_1 - x_2) \preccurlyeq \tilde{0}$.*

Proof. Let $\tilde{f}$ be quasiconvex, and let $x_1, x_2 \in \Omega$ be such that $\tilde{f}(x_1) \preccurlyeq \tilde{f}(x_2)$. Since $\tilde{f}$ is differentiable at x_2, we have $f_*(.,\alpha)$ and $f^*(.,\alpha)$ are differentiable at x_2 for each $\alpha \in [0,1]$. Hence

$$f_*(\lambda x_1 + (1-\lambda)x_2, \alpha) - f_*(x_2,\alpha) = \lambda \nabla f_*(x_2,\alpha)^t(x_1 - x_2) + \lambda \|x_1 - x_2\| \theta_*(x_2, \lambda(x_1 - x_2), \alpha), \tag{21}$$

$$f^*(\lambda x_1 + (1-\lambda)x_2, \alpha) - f^*(x_2,\alpha) = \lambda \nabla f^*(x_2,\alpha)^t(x_1 - x_2) + \lambda \|x_1 - x_2\| \theta^*(x_2, \lambda(x_1 - x_2), \alpha), \tag{22}$$

where as $\lambda \to 0, \theta_*(x_2, \lambda(x_1 - x_2), \alpha) \to 0, \theta^*(x_2, \lambda(x_1 - x_2), \alpha) \to 0$.

By the quasiconvexity of $\tilde{f}$, and as $\tilde{f}(x_1) \preccurlyeq \tilde{f}(x_2)$, we have

$$\tilde{f}[\lambda x_1 + (1-\lambda)x_]] \preccurlyeq \max\{\tilde{f}(x_1), \tilde{f}(x_2)\} = \tilde{f}(x_2) \quad \text{for each } \lambda \in (0,1) \tag{23}$$

that is, for each $\alpha \in [0,1]$,

$$f_*(\lambda x_1 + (1-\lambda)x_2, \alpha) \leqslant f_*(x_2,\alpha) \tag{24}$$

and

$$f^*(\lambda x_1 + (1-\lambda)x_2, \alpha) \leqslant f^*(x_2,\alpha). \tag{25}$$

Now (21), (22), (24) and (25) imply that

$$\lambda \nabla f_*(x_2,\alpha)^t(x_1 - x_2) + \lambda \|x_1 - x_2\| \theta_*(x_2, \lambda(x_1 - x_2), \alpha) \leqslant 0 \tag{26}$$

and

$$\lambda \nabla f^*(x_2,\alpha)^t(x_1 - x_2) + \lambda \|x_1 - x_2\| \theta^*(x_2, \lambda(x_1 - x_2), \alpha) \leqslant 0 \tag{27}$$

Dividing (26) and (27) by λ and taking $\lambda \to 0$, we get

$$\nabla f_*(x_2,\alpha)^t(x_1 - x_2) \leqslant 0, \tag{28}$$

$$\nabla f^*(x_2,\alpha)^t(x_1 - x_2) \leqslant 0. \tag{29}$$

This is equivalent to

$$\tilde{\nabla} \tilde{f}(x_2)^t(x_1 - x_2) \preccurlyeq \tilde{0}. \tag{30}$$

Conversely, suppose that $x_1, x_2 \in \Omega$ and $\tilde{f}(x_1) \preccurlyeq \tilde{f}(x_2)$. We need to show that given part 1, for each $\alpha \in [0,1]$, the inequalities (24) and (25) are true. Let

$$P = \{x : x = \lambda x_1 + (1-\lambda)x_2, \lambda \in (0,1), \tilde{f}(x) \succ \tilde{f}(x_2)\}.$$

We show that P is empty. If not, then suppose that there exists an $y \in P$. Let $y = \lambda x_1 + (1-\lambda)x_2$, for some $\lambda \in (0,1)$ and $\tilde{f}(y) \succ \tilde{f}(x_2)$. So there exists $\alpha_0 \in [0,1]$ such that either $f_*(y,\alpha_0) > f_*(x_2,\alpha_0)$ or $f^*(y,\alpha_0) > f_*(x_2,\alpha_0)$. Without loss of generality assume that $f_*(y,\alpha_0) > f_*(x_2,\alpha_0)$. Now since $\tilde{f}$ is differentiable, $f_*(.,\alpha)$ is also differentiable and in particular $f_*(.,\alpha_0)$ is differentiable and hence continuous. Therefore, there exists a $\delta \in (0,1)$ such that

$$f_*(ky + (1-k)x_2, \alpha_0) > f_*(x_2,\alpha_0) \quad \forall k \in [\delta, 1] \tag{31}$$

and

$$f_*(y,\alpha_0) > f_*(\delta y + (1-\delta)x_2, \alpha_0). \tag{32}$$

By (32) and the mean value theorem, we must have

$$0 < f_*(y,\alpha_0) - f_*(\delta y + (1-\delta)x_2,\alpha_0) = (1-\delta)\nabla f_*(w,\alpha_0)^t(y - x_2), \tag{33}$$

where $w = k'y + (1-k')x_2$ for some $k' \in (\delta, 1)$.

Since $y = \lambda x_1 + (1-\lambda)x_2$ and $(1-\delta) > 0$, (33) gives

$$0 < \nabla f_*(w,\alpha_0)^t(x_1 - x_2). \tag{34}$$

From (31), we can obtain $f_*(w,\alpha_0) > f_*(x_2,\alpha_0)$.

But since $f_*(w,\alpha_0) > f_*(x_2,\alpha_0) > f_*(x_1,\alpha_0)$ and w is a convex combination of y and x_2 and hence in the convex combination of x_1 and x_2, say $w = \lambda_1 x_1 + (1-\lambda_1)x_2$, where $\lambda_1 \in (0,1)$.

Now by the assumption of the theorem

$$\tilde{\nabla}(w)^t(x_1 - w) \leqslant 0 \Rightarrow (1-\lambda_1)\tilde{\nabla}(w)^t(x_1 - x_2) \leqslant 0 \tag{35}$$

and therefore, we must have

$$0 \geqslant \nabla f_*(w,\alpha_0)^t(x_1 - w) = (1-\lambda_1)\nabla f_*(w,\alpha_0)^t(x_1 - x_2). \tag{36}$$

The inequality (36) is not compatible with (34).

Hence P is empty, and the proof is complete. □

Definition 4.15 (*Pseudoconvex fuzzy mapping*). Let $\tilde{f} : \Omega \subseteq \mathbb{R}^n \to \mathscr{F}$ be a fuzzy mapping, where Ω is a nonempty convex set in $\mathbb{R}^n$ and $\tilde{f}$ differentiable on Ω. The mapping $\tilde{f}$ is said to be pseudoconvex if for each $x_1, x_2 \in \Omega$ with $\tilde{0} \preccurlyeq \nabla\tilde{f}(x)^t(y-x)$ we have $\tilde{f}(x) \preccurlyeq \tilde{f}(y)$. The fuzzy mapping $\tilde{f}$ is said to be pseudoconcave if $-\tilde{f}$ is pseudoconvex.

Theorem 4.16. *Let $\tilde{f} : \Omega \subseteq \mathbb{R}^n \to \mathscr{F}$ be a differentiable pseudoconvex fuzzy mapping, where Ω is a nonempty open convex set in $\mathbb{R}^n$. Then $\tilde{f}$ is strictly quasiconvex.*

Proof. By contradiction, suppose that $\exists x, y \in \Omega$ such that $\tilde{f}(x) \neq \tilde{f}(y)$, ($\tilde{f}(x)$ and $\tilde{f}(y)$ are comparable) and $\tilde{f}(z) \succcurlyeq \max\{\tilde{f}(x), \tilde{f}(y)\}$, where $z = \lambda x + (1-\lambda)y$ for some $\lambda \in (0,1)$. Without loss of generality let $\tilde{f}(x) \prec \tilde{f}(y)$, so that

$$\tilde{f}(x) \prec \tilde{f}(y) \preccurlyeq \tilde{f}(z). \tag{37}$$

Since $\tilde{f}$ is pseudoconvex we have $\nabla\tilde{f}(z)^t(x-z) \prec \tilde{0}$. Now since $\nabla\tilde{f}(z)^t(x-z) \prec \tilde{0}$ and $x - z = -\frac{1-\lambda}{\lambda}(y-z)$, thus $\tilde{0} \prec \nabla\tilde{f}(z)^t(y-z)$, and by pseudoconvexity of $\tilde{f}$, we must have $\tilde{f}(z) \preccurlyeq \tilde{f}(y)$. From (37), we get $\tilde{f}(z) = \tilde{f}(y)$.

Also, since $\tilde{0} \prec \nabla\tilde{f}(z)^t(y-z)$, $\exists$ a point $w = kz + (1-k)y$ with $k \in (0,1)$ such that

$$\tilde{f}(y) = \tilde{f}(z) \prec \tilde{f}(w). \tag{38}$$

Again by pseudoconvexity of $\tilde{f}$, we have $\nabla\tilde{f}(w)^t(y-w) \prec \tilde{0}$. Similarly, $\nabla\tilde{f}(w)^t(z-w) \prec \tilde{0}$. That is, we must have

$$\nabla\tilde{f}(w)^t(y-w) \prec \tilde{0} \quad \text{and} \quad \nabla\tilde{f}(w)^t(z-w) \prec \tilde{0}.$$

Now as $y - z = \frac{k}{1-k}(w - z)$ and hence the above two inequalities can not be satisfied together. This contradiction shows that $\tilde{f}$ is strictly quasiconvex. □

Theorem 4.17. *Let $\tilde{f} : \Omega \to \mathscr{F}$ be a pseudoconvex fuzzy mapping, where Ω is a nonempty open convex set in $\mathbb{R}^n$. Let $\tilde{f} \in \mathscr{E}$. Then $x \in \Omega$ is a point of global minimum if and only if $\tilde{\nabla}\tilde{f}(x) = \tilde{0}$.*

Proof. If x is a point of global minimum then by Theorem 3.13 $\tilde{\nabla}\tilde{f}(x) = \tilde{0}$.

Conversely, suppose that $\tilde{\nabla}\tilde{f}(x) = \tilde{0}$. Then we have $\tilde{\nabla}\tilde{f}(x)^t(y - x) = \tilde{0}\ \forall y \in \Omega$. Hence, by the pseudoconvexity of $\tilde{f}$, we have $\tilde{f}(x) \preccurlyeq \tilde{f}(y)$ for all $y \in \Omega$. Hence, x is a point of global minimum.

Wu [15] has given saddle point optimality conditions in fuzzy optimization problem, Nanda and Kar [10] have studied convex fuzzy mappings and some of its application to optimization, but differentiability concept is not taken into consideration. Syau [14] have discussed differentiability and convex fuzzy mappings. But constrained fuzzy optimization under differentiability has not been discussed in the literature so far. □

5. Constrained fuzzy minimization problems

Let Ω_0 be an open set in $\mathbb{R}^n$, and let $\tilde{f} : \Omega_0 \to \mathscr{F}$ be a fuzzy mapping. Let $\tilde{g} : \Omega_0 \to \mathscr{F}^m$ be an m-dimensional fuzzy function.

A constrained fuzzy minimization problem is

$$\begin{aligned} &\text{Minimize} \quad \tilde{f}(x) \\ &\text{subject to} \quad \tilde{g}(x) \preccurlyeq \tilde{0}, \\ &\qquad\qquad\quad x \geqslant 0, \end{aligned} \tag{39}$$

where $x \in \mathbb{R}^n$.

Here, "Minimize $\tilde{f}(x)$" is as explained in Section 3, and for $\tilde{g} = (\tilde{g}_i)_{i=1}^m$, (39) means ($\tilde{g}_i(x) \preccurlyeq \tilde{0}$ for each $i = 1, \ldots, m$).

5.1. The Kuhn–Tucker stationary point fuzzy problem (KTFP)

Let $\tilde{f}$ and $\tilde{g}$ respectively, be a fuzzy mapping and an m-dimensional fuzzy vector function, both defined on Ω_0.

Define the Lagrangian fuzzy function as

$$\tilde{L}(x, u) = \tilde{f}(x) + u^t\tilde{g}(x).$$

The Kuhn–Tucker stationary point fuzzy problem (KTFP) is to find $x \in \Omega_0, u \in \mathbb{R}^m$ if they exist, such that

$$\tilde{\nabla}\tilde{f}(x) + u^t\tilde{\nabla}\tilde{g}(x) = \tilde{0}, \tag{40}$$

$$\tilde{g}(x) \preccurlyeq \tilde{0}, \tag{41}$$

$$u^t\tilde{g}(x) = \tilde{0^*}, \tag{42}$$

$$u \geqslant 0. \tag{43}$$

Note that in (40) $\tilde{0} \in \mathscr{F}$, in (41) $\tilde{0} \in \mathscr{F}^m$ and in (42) $\tilde{0^*} \in \mathscr{F}$ is such that $\operatorname{core}(\tilde{0^*}) = 0$.

Also (40) is equivalent to

$$\nabla f_*(x, \alpha) + u^t\nabla g_*(x, \alpha) = 0, \tag{44}$$

$$\nabla f^*(x, \alpha) + u^t\nabla g^*(x, \alpha) = 0 \tag{45}$$

for each $\alpha \in [0, 1]$.

Theorem 5.1 (Sufficient optimality criteria for the KTFP).

Let $\hat{x} \in \Omega_0 \subseteq \mathbb{R}^n$, let Ω_0 be open, and let $\tilde{f}$ and $\tilde{g}$ be differentiable and convex at $\hat{x}$. Let $\tilde{f} \in \mathscr{E}$ and the Lagrangian function $\tilde{L}(x, u) \in \mathscr{E}$ (a comparable function of x). If $(\hat{x}, \hat{u})$ is a solution of KTFP, then $\hat{x}$ is a solution of FMP.

Proof. Since $\tilde{f}$ is convex and differentiable at $\hat{x}$, we have both $f_*(x,\alpha)$ and $f^*(x,\alpha)$ are convex and differentiable at $\hat{x}$ for each $\alpha \in [0,1]$. Now for any $x \in \Omega$, $\alpha = 1$, we have by Theorem 4.3

$$\begin{aligned} f_*(x,1) - f_*(\hat{x},1) &\geqslant \nabla f_*(\hat{x},1)(x-\hat{x}),\\ &= -\hat{u}^t \nabla g_*(\hat{x},1)(x-\hat{x}) \quad \text{(by (44))},\\ &\geqslant \hat{u}^t[g_*(\hat{x},1) - g_*(x,1)] \quad (\text{since } \tilde{g} \text{ is convex}),\\ &= -\hat{u}^t g_*(x,1) \quad \text{(by (42))},\\ &\geqslant 0 \quad (u \geqq 0 \text{ and } g_*(x,1) = 0). \end{aligned}$$

Since $\tilde{f} \in \mathscr{E}$, we conclude that $\hat{x}$ is a solution to FMP. $\square$

The following example justify Theorem 5.1.

Example 5.2

$$\begin{aligned} &\text{Min} && \tilde{f}(x,y) = \tilde{a}x^2 + \tilde{b}y^2\\ &\text{subject to} && \tilde{h}(x,y) = \tilde{c}(x-2)^2 + \tilde{d}(y-2)^2 \leqq \tilde{k},\\ &&& x>0,\ \ y>0, \end{aligned} \tag{46}$$

where $\tilde{a} = \langle 0;2;4\rangle$, $\tilde{b} = \langle 0;2;4\rangle$, $\tilde{c} = \langle 0;2;4\rangle$, $\tilde{d} = \langle 0;2;4\rangle$, $\tilde{k} = \langle 0;2;4\rangle$.

Note that (46) is equivalent to:

$$2\alpha(x-2)^2 + 2\alpha(y-2)^2 \leqslant 2\alpha, \tag{47}$$

$$(4-2\alpha)(x-2)^2 + (4-2\alpha)(y-2)^2 \leqslant 4-2\alpha \tag{48}$$

for each $\alpha \in [0,1]$. That is same as

$$2\alpha(x-2)^2 + 2\alpha(y-2)^2 - 2\alpha \leqslant 0, \tag{49}$$

$$(4-2\alpha)(x-2)^2 + (4-2\alpha)(y-2)^2 - (4-2\alpha) \leqslant 0 \tag{50}$$

for each $\alpha \in [0,1]$.

Now consider the function $\tilde{g} = \langle g_*(x,\alpha), g^*(x,\alpha)\rangle$ where

$$\begin{aligned} &g_*(x,\alpha) = 2\alpha(x-2)^2 + 2\alpha(y-2)^2 - 2\alpha \quad \text{and}\\ &g^*(x,\alpha) = (4-2\alpha)(x-2)^2 + (4-2\alpha)(y-2)^2 - (4-2\alpha). \end{aligned}$$

It is easy to see that $\tilde{g}$ is a fuzzy mapping.

Now let $\tilde{L}((x,y),u) = \tilde{f}(x,y) + u^t\tilde{g}(x,y)$, then

$$\begin{aligned} &L_*((x,y),u,\alpha) = 2\alpha x^2 + 2\alpha y^2 + u[2\alpha(x-2)^2 + 2\alpha(y-2)^2 - 2\alpha],\\ &L^*((x,y),u,\alpha) = (4-2\alpha)x^2 + (4-2\alpha)y^2 + u(4-2\alpha)[(x-2)^2 + (y-2)^2 - 1],\\ &\nabla_{(x,y)}L_*((x,y),u,\alpha) = (4\alpha x + 4u\alpha(x-2), 4\alpha y + 4u\alpha(y-2)),\\ &\nabla_{(x,y)}L^*((x,y),u,\alpha) = (2(4-2\alpha)x + 2u(4-2\alpha)(x-2), 2(4-2\alpha)y + 2u(4-2\alpha)(y-2)). \end{aligned}$$

Now we have to solve

$$\nabla_{(x,y)}L_*((x,y),u,\alpha) = 0 = \nabla_{(x,y)}L^*((x,y),u,\alpha), \tag{51}$$

$$\tilde{g}(x,y) \preccurlyeq \tilde{0}, \tag{52}$$

$$u\tilde{g}(x,y) = \tilde{0}, \tag{53}$$

$$u \geqslant 0. \tag{54}$$

That is to solve

$$4\alpha x + 4u\alpha(x-2) = 0 = 4\alpha y + 4u\alpha(y-2), \tag{55}$$

$$2(4-2\alpha)x + 2u(4-2\alpha)(x-2) = 0 = 2(4-2\alpha)y + 2u(4-2\alpha)(y-2), \tag{56}$$

$$2\alpha(x-2)^2 + 2\alpha(y-2)^2 - 2\alpha \leqslant 0, \tag{57}$$

$$(4-2\alpha)(x-2)^2 + (4-2\alpha)(y-2)^2 - (4-2\alpha) \leqslant 0, \tag{58}$$

$$u[2\alpha(x-2)^2 + 2\alpha(y-2)^2 - 2\alpha] = 0, \tag{59}$$

$$u[(4-2\alpha)(x-2)^2 + (4-2\alpha)(y-2)^2 - (4-2\alpha)] = 0, \tag{60}$$

$$u \geqslant 0. \tag{61}$$

Solving (55)–(61), we get $x = 2u/(u+1) = y$ and $u = 2\sqrt{2} - 1$. Thus, $x = 2 - \frac{1}{\sqrt{2}} = y$.

Thus, the minimum value of the problem is found to be $\tilde{a}(2 - \frac{1}{\sqrt{2}})^2 + \tilde{b}\left(2 - \frac{1}{\sqrt{2}}\right)^2$.

6. Conclusion

The concept of convex fuzzy mappings without differentiability has been discussed in the literature by many researchers. The objective of this paper is to introduce the concept of convex fuzzy mappings and generalized convex fuzzy mappings under differentiability. Using this concept the sufficient optimality condition for constrained fuzzy minimization problem has been derived in Section 5. However, different types of necessary and sufficient optimality conditions such as Fritz John constraint qualification and Slater's constraint qualification for fuzzy nonlinear optimization problem can be derived in a similar way which is the future research scope of this paper. Also other types of generalized convexity such as invexity and bonvexity for fuzzy mappings with differentiability can be defined in a similar way.

Acknowledgement

The authors thank the referees for their valuable suggestions which improved the presentation of the paper.

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